POWER OSCILLATIONS AND LIMIT CYCLES IN FISSION REACTORS WITH AND WITHOUT DELAYED NEUTRONS

by

A. K. SINHA

NET P
1989
M
SIN
PCW

NUCLEAR ENGINEERING AND TECHNOLOGY PROGRAMME

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

December 1989

POWER OSCILLATIONS AND LIMIT CYCLES IN FISSION REACTORS WITH AND WITHOUT DELAYED NEUTRONS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

by

A. K. SINHA

to the

NUCLEAR ENGINEERING AND TECHNOLOGY PROGRAMME
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

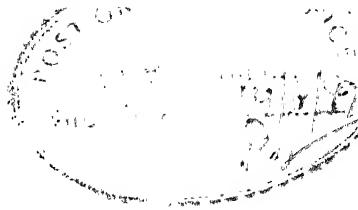
December 1989

5 APR 1990

CENTRAL LIBRARY
I. I. T., KANPUR

Acc. No. A107894

NETP-1989-M-SIN-POW



CERTIFICATE

This is to certify that this work entitled, **POWER OSCILLATIONS AND LIMIT CYCLES IN FISSION REACTORS WITH AND WITHOUT DELAYED NEUTRONS**, has been carried out by Amar Kumar Sinha under our supervision and has not been submitted elsewhere for a degree.

K. Sri Ram

(K. Sri Ram)
Professor
Dept. of Nuclear Engineering
Indian Institute of Technology
Kanpur 208 016

M S Kalra

(M. S. Kalra)
Assistant Professor
Dept. of Nuclear Engineering
Indian Institute of Technology
Kanpur 208 016

ABSTRACT

Power and fuel temperature excursions in a fission reactor are analyzed using two-temperature model with and without delayed neutrons. The governing differential equations are recast in a dimensionless form for theoretical studies. The resulting system of differential equations is analyzed using Hopf bifurcation theorem. It is found that depending upon the reactor kinetics and thermal hydraulic parameters, the conditions of Hopf bifurcation theorem may or may not be satisfied. The parametric equations for the bifurcation line or curve are obtained and plotted. The presence of Hopf bifurcation confirms the existence of periodic orbits in a finite neighborhood of the operating point, for the appropriate values of the parameters as ascertained from the bifurcation curve. Stable limit cycles for both with and without delayed neutrons are calculated numerically, showing sustained nonlinear periodic oscillations in reactor power, fuel and coolant temperatures, and the precursor concentration. The time periods and the amplitudes of the oscillations are also obtained numerically.

It is found that for the conditions of Hopf bifurcation theorem to be satisfied, the coolant temperature coefficient must be negative. Fuel temperature coefficient may be positive or negative. In both cases, stable limit cycles exist.

CONTENTS

NOMENCLATURE

P	reactor power, W
C	delayed neutron precursors (in power units), W
P_o, C_o	operating or steady state values of P and C, W
$\mathbb{P}$	dimensionlelss power [= $(P - P_o)/P_o$]
$\mathbb{C}$	dimensionless delayed neutron precursors [= $(C-C_o)/C_o$]
T_f	average fuel temperature, ^oK
T_i, T_e	inlet & exit temperatures of the coolant, ^oK
T_c	average coolant temperature [= $(T_e+T_i)/2$], ^oK
T_f^o, T_c^o	operating or steady state values of T_f and T_c, ^oK
$\mathbb{T}_f$	dimensionless fuel temperature [= $(T_f-T_f^o)/(T_f^o-T_i)$]
$\mathbb{T}_c$	dimensionless coolant temperature [= $(T_c-T_c^o)/(T_c^o-T_i)$]
l	neutron life time, sec.
C_f, C_c	fuel and coolant heat capacities, JK^{-1}
a_f =	$\alpha_f (T_f^o-T_i)/(\lambda\, l)$, dimensionless
a_c =	$\alpha_c (T_c^o-T_i)/(\lambda\, l)$, dimensionless
b =	$\beta/(\lambda\, l)$, dimensionless
c =	C_f/C_c, dimensionless
p =	$P_o/[\lambda\, C_f (T_f^o-T_c^o)]$, dimensionless
θ =	$(T_c^o-T_i)/(T_f^o-T_i)$, dimensionless
α_f, α_c	fuel and coolant temperature coefficients of reactivities, K^{-1}
β	delayed neutron fraction, dimensionless
λ	delayed neutron decay constant, sec^{-1}
ρ	reactivity, dimensionless

τ	dimensionless time (= λ t)
H_f	fuel to coolant heat transfer coefficient (WK^{-1})
H_c	Twice the product of the mass flow rate and specific heat of the coolant (WK^{-1})

CHAPTER I

INTRODUCTION

In this chapter first of all the subject of reactor dynamics, in as much as it is relevant to the present study, will be introduced. This will be followed by a review of some papers in the related field and finally the present work will be outlined.

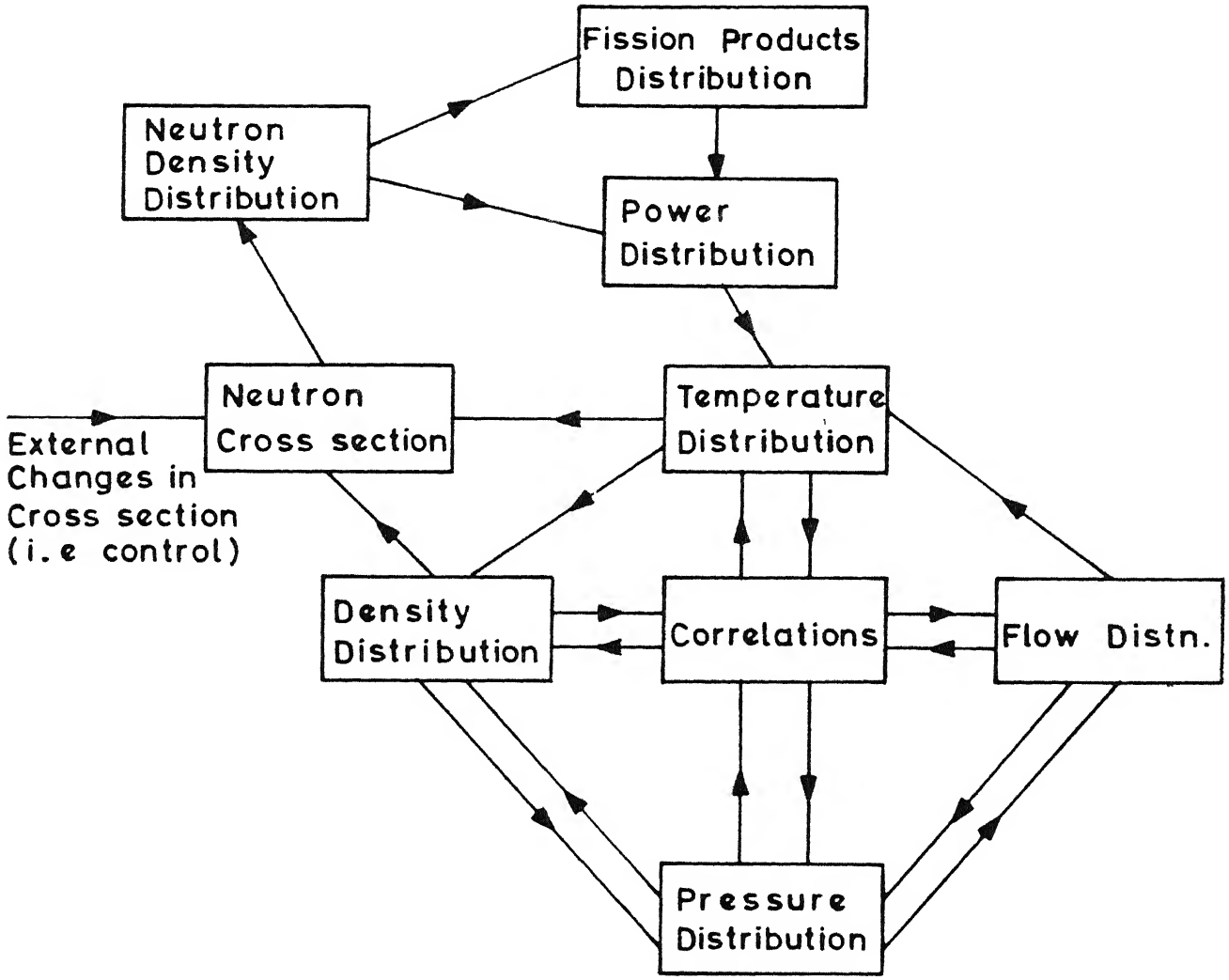
1.1 PROLOGUE

To understand the dynamic behavior of a nuclear reactor, or more precisely the core of a nuclear reactor, one must know at any instant of time the detailed spatial distribution of the neutrons, the rate of power production, the temperature and density of all the materials present in the core, the pressure distribution, and the flow rates of all the materials present in the fluid or gaseous phases. The importance of this knowledge can not be over estimated since predictions of the core integrity and the reactor safety as well as control depend on how accurately these distributions are known. Reactor dynamic calculations are concerned mainly with estimating these distributions with reasonable accuracy. It must be pointed out that the amount of fissile materials and the amount of fission products in currently operating 1000 MW reactors are such that if a severe accident were to take place leading to a core meltdown and release of the fission products to the environment, grave consequences to the population might result. Thus careful consideration is required

in order to develop methods in reactor dynamics which can predict the core behavior with sufficient accuracy and reliability.

To be able to predict the dynamic behavior of physical systems, an appropriate mathematical model has to be developed. The model should be a true description of the system or the best estimate of the true description. However, the equations describing the model have to be of a form and level of complexity that can be solved with available resources. These requirements contradict each other, and therefore the success of any dynamic model in predicting the future of system behavior is strongly dependent on the assumptions made in formulating the model and the problem to which model is applied.

The core dynamic process can be represented by the flow chart shown. A change in the neutron flux or neutron density distribution will cause a change in the reactor power distribution which in turn will produce changes in the temperature and pressure distributions. As a result, the density of the materials involved, namely, the fuel and the coolant, will change. This includes possible phase transitions and the onset of processes such as coolant boiling. As a result, the fluid flow rate, both in magnitude and direction will also be affected. Processes such as fuel expansion, meltdown, the fuel coolant interaction might also be initiated. Changes in the state variables affect the various thermal and hydraulic correlations and the neutron cross sections of the materials involved. These in turn affect the neutron distribution and so on. Appropriate justification is required to break any of the links in the model. Since



A FLOWCHART OF THE RELATIONSHIP BETWEEN PROCESSES INVOLVED IN REACTOR DYNAMICS CALCULATIONS.

the links do exist in the physical reality the development of a full and precise model of all the processes presented in Flow Chart is not feasible. Such a model contains many complicated equations in a rather complex three dimensional geometry. In the formulation of a practical solution, therefore, several of the processes are usually either neglected or approximated by simplified models. For example, in many instances the effect of temperature and density changes on the reactor cross sections can be approximated by feedback reactivity coefficients.

When one is faced with the problem of solving reactor dynamic equations, i.e. to find the core power, temperature (fuel, coolant, moderator), flux, enthalpy and flow distributions as a function of time, many questions arise and must be answered prior to choosing the method of solution and even computer programs to be used. Some of these questions are listed below which are obviously not complete.

1. What approximation to the Boltzmann equation should be chosen - the neutron transport equation, the diffusion equation, or the lumped parameters kinetic equations?
2. Should the problem be solved in three, two or one spatial dimension, or should one use the lumped parameter model represented by the point kinetic equations in order to obtain the change in time of the power distribution?
3. What approximation, or rather how many energy groups, should one use to represent the neutron energy range? Solution range from a single group to few (2 to 6) energy groups and to large number (30) of energy groups.

4. What is the required time span of the solution? Some transients last a small fraction of a second; others last seconds to minutes before the effects are seen; still others may last several hours. Each of these probably requires a completely different solution scheme.
5. How should one account for the significant changes that take place in the neutron cross sections used in the diffusion or transport approximations? In other words, how does one account for the feedback due to temperature and density changes in the core?

Based on the type of reactor under consideration, simplifying assumptions and approximations, we end up with a set of equations which in general are nonlinear. A linear analysis of such a dynamical system is useful only when disturbance is small from the operating point since in this case only linear terms are dominant, nonlinear terms having insignificant contribution. However, if the perturbation is large, that is disturbance is large from the operating point, it is the nonlinear terms which dominate. Therefore, in order to have a better understanding of the reactor dynamics, we have to look for nonlinear analysis and bifurcation theory using state space concepts. Linear analysis shows only stable and unstable fixed points, that too, only locally but nonlinear analysis may show orbits (such as limit cycles which may be stable or unstable) which makes it much more interesting. Apart from limit cycles (periodic orbits) nonlinear analysis may also show quasi periodic, aperiodic or chaotic orbits. By studying bifurcation we may find out whether such orbits occur in

a particular dynamical system and if so then for what values of parameters or what ranges of parameters.

From control and safety point of view, if for certain values of parameters, a disturbance from steady state point is going to take it to limit cycle i.e., ending up with continuous oscillation in power, fuel temperature, coolant temperature or precursor concentration, then such a phenomenon is certainly worth studying. A limit cycle is an isolated periodic solution of an autonomous system represented in the state space by an isolated closed path. The neighboring paths are not closed, but spiral into or away from the limit cycle. The study of such systems using Hopf bifurcation theorem provides us with a clue to existence of limit cycles. In Hopf bifurcation, a pair of complex conjugate eigen values of the Jacobian matrix crosses the imaginary axis with non zero velocity, and makes it possible for the limit cycles to exist.

1.2 TIME DEPENDENT REACTOR KINETICS EQUATIONS

The neutron kinetics equations, sometimes designated as the point kinetics equations, are the most frequently encountered equation in reactor dynamics. They can be obtained from basic principles as is done in [14, 27, 28]. They can also be precisely derived from the transport or diffusion equation as has been done in [26].

1.2.1 Space Dependent Dynamic Model

The time dependent diffusion equation for neutrons in a reactor as given in [31] is

$$\frac{\partial N}{\partial t} = Dv\,\nabla^2 N - \Sigma_a\, v\, N + S \qquad (1.1)$$

where

$N(r,t)\,dV$ = number of neutrons in a volume element dV at a point r at time t;

$Dv\nabla^2 N\,dV$ = number of neutrons diffusing into dV per unit time at time t;

$\Sigma_a vN\,dV$ = number of neutrons absorbed in dV per unit time at time t;

$S(r,t)\,dV$ = number of neutrons produced in dV per unit time at time t;

D = diffusion constant;

Σ_a = macroscopic neutron absorption cross section;

v = neutron speed.

Here two assumptions have been made. They are:

(a) Neutron current density obeys Fick's law, i.e.,

$$J = - D\, v\, \nabla N \qquad (1.2)$$

(b) All the coefficients appearing in (1.1) are position independent that is their numerical values are essentially averages over the neutron velocity distribution.

Writing down diffusion equation and delayed neutron equation as given by Hetrick [31], we have

$$\frac{\partial N}{\partial t} = D\, v\, \nabla^2 N - \Sigma_a\, v\, N + (1-\beta)\, k_\infty\, \Sigma_a\, v\, N + \Sigma_i\, \lambda_i\, C_i + S_o \qquad (1.3)$$

$$\frac{\partial C_i}{\partial t} = \beta_i\, k_\infty\, \Sigma_a\, v\, N - \lambda_i\, C_i \qquad (1.4)$$

where,

k_∞ = infinite medium reproduction factor;

β_i = delayed neutron fraction for the i-th precursor;

$\beta = \sum_i \beta_i$;

λ_i = decay constant of i-th precursor;

$C_i(r,t)$ = Volumetric density of the i-th of precursor;

$S_o(r,t)$ = term representing sources of neutrons different from the fission processes.

Equations (1.3) and (1.4) constitute a set of coupled partial differential equations. They together form an infinite dimensional dynamical system. Fortunately under certain assumptions this system of infinite dimensions can be reduced to a finite dimensional system, i.e., a set of ordinary differential equations. This can be achieved if we use the lumped parameter or point reactor model.

1.2.2 The Point Reactor Model

The fundamental assumption underlying this model is that all except the fundamental spatial harmonics of the flux, following a flux disturbance, have been attenuated in a very short time (a few micro seconds) compared to times of reactor kinetics of interest as pointed out by Ash [14]. In other words the density of neutrons in the reactor is assumed to rise or fall as a whole following a change in reactivity. This means all spatial dependences factor away, leaving only time as the independent variable.

Substituting as in [30],

$$N(r,t) = f(r)\, n(t);\quad C_i(r,t) = g_i(r)\, C_i(t) \qquad (1.5)$$

in Eqs. (1.3) - (1.4) and forcing the condition that space dependence from the resulting equations must be removed it is required that f/g_i, $\nabla^2 f/f$ and S_o/f be independent of position. Assuming

$$g_i = f;\quad \nabla^2 f + B^2 f = 0$$

and letting

$$q(t) = S_o(r, t)/f(r)$$

we end up with the point reactor model as

$$\frac{dn}{dt} = \frac{\rho-\beta}{l} n + \sum_i \lambda_i C_i + q \qquad (1.6)$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{l} n - \lambda_i C_i \qquad (1.7)$$

where,

B^2 = fundamental buckling,

ρ = reactivity,

l = neutron generation time,

q = effective source strength.

Eqs. (1.6) - (1.7) can be further simplified by assuming

(a) Reactor is operating at high power,

(b) All the delayed neutrons have a single mean life time.

First condition may be used to drop q in (1.6) [30] and the second condition implies $\beta_i = \beta$. Then we get the simplified equations.

$$\frac{dn}{dt} = \frac{\rho-\beta}{l} n + \lambda C \qquad (1.8)$$

$$\frac{dC}{dt} = \frac{\beta}{l} n - \lambda C \qquad (1.9)$$

Eqs. (1.8)-(1.9) are the simplified point kinetic model which is a system of two coupled differential equations. All the assumptions made in arriving at these equations have been clearly stated at appropriate places.

1.3 LITERATURE REVIEW

In one of the important papers, Popov [25] has shown the effect of delayed neutrons on the stability of the reactor. He concludes that if stability can be proved by a particular type of Liapunov function without considering the effect of delayed neutrons, then the system is asymptotically stable when the effect of the delayed neutrons is taken into account. Baran and Meyer [22] point out that Liapunov function considered by Popov was of a very particular type, and therefore, one cannot assume in general that delayed neutrons have a stabilizing effect. Further, he gives an example to show that stability without delayed neutrons does not imply stability with delayed neutrons. He has also derived a new criterion for stability in the large. Gyftopoulos et al [32] have shown by considering Lyapunov's second method that if a nuclear reactor is stable for arbitrary small perturbation without delayed neutrons, it is also stable for arbitrary small perturbation with delayed neutrons. Smets [23] studies linearized reactor dynamics equations and shows that delayed neutrons do not always improve the stability of nuclear reactors and that a reactor may be unstable although it is stable when delayed

neutrons are neglected. Devooght et al [7] discuss four methods of analysis of nonlinear stability that can be used when the reactors are unstable above a certain equilibrium power. He further asserts that topological methods and Liapunov's second method may not be applicable for complex systems. For complex systems he describes two methods one by Aizermann and second due to Rosen. Akcasu and Shotkin [2] studied bounded periodic behavior of the reactor power for the cases when the equilibrium power is greater than the critical power level in the absence of delayed neutrons. For such reactors he also derives expressions for amplitude and frequency of the limit cycles. The technique they have employed is essentially perturbation method. They have also been able to derive a describing function which plays important role in controls. Shotkin et al [3] have considered reactors with feedback linearly proportional to the power level and shown that consideration of delayed neutrons permit the existence of unstable limit cycles. Technique adopted by them is essentially the same as [2].

Vreeke and Sandquist [8] present a generalized approach to phase space analysis of reactor kinetics equations. Their method of approach employs a digital computer and a graphics terminal to construct rapidly and accurately two-dimensional and perspective three-dimensional phase space portraits over the region of interest. In this paper they have considered phase space solutions of second and third order systems of equations. The method is demonstrated on three different nonlinear problem of interest to nuclear reactor kinetics. A second order problem with temperature dependent reactivity and two third-order problems with

reactivity a function of two effective temperatures and ^{135}Xe concentration are considered. Schmidt and Hetrick [4] use Fourier series methods for determining the frequency and fundamental amplitudes of the oscillations in reactivity and power level. They also examine the system for the effect of delayed neutrons on oscillatory solutions and conclude that when delayed neutrons are included, unstable limit cycles as well as stable limit cycles are found. Poddar et al [5] have examined oscillatory behavior in nuclear reactors ith two temperature feedback using perturbation method, but without delayed neutrons. They establish necessary and sufficient conditions for asymptotic stability under oscillatory conditions. Poddar and Trasi [9] have pointed out some discrepancy in [34] and have modified conclusions of [34]. By using both Welton's criterion as well as Liapunov's second method, the system considered in [34] can only be shown to be asymptotically stable in the small and in the large when the delayed temperature coefficient lies between certain limits, which include a range of negative and positive values. Gupta and Trasi [10] have, for a Newtonian feedback and a step input of reactivity, solved point reactor kinetics equations using perturbation techniques in the presence of delayed neutrons. They have shown that asymptotic power level is independent of perturbation modes higher than the first and that the change in the steady power level is inversely proportional to the magnitude of the reactivity coefficient of fuel temperature. Ward and Lee [6] use singular perturbation method for the analysis of large power oscillations in nuclear reactors. Such large oscillations occur as the system reactivity becomes significantly positive and

then negative. This paper introduces a potential method for the direct limit cycle solution, in phase space. Keener and Cohen [1] have shown existence of limit cycle in two-temperature feedback model without delayed neutrons using perturbation techniques. Ashok Kumar [17] has analysed the two-temperature feedback model with delayed neutrons. He uses a perturbation approach for linearization and establishes the conditions for oscillatory and non oscillatory transient behavior as well as sustained oscillations, and verified the results of this linear analysis by numerical computation.

In most of the works just described presence of delayed neutrons have been considered only for stability analysis. Limit cycle analysis has been done by different techniques neglecting the presence of delayed neutrons. Manmohan Pandey [24] has shown the existence of stable limit cycles in two temperature feedback model neglecting the presence of delayed neutrons by using Hopf bifurcation theorem. However, he considered only positive fuel temperature coefficient of reactivity.

1.4 PRESENT WORK

The purpose of the present study is to analyze the nonlinear system of equations governing reactor power and temperatures, without resorting to linearization or perturbation methods. This ensures that the predicted behavior will be observed in the actual nonlinear system and will be valid for arbitrary disturbances.

The model used is the two-temperature feedback model, with and without delayed neutrons. For each of the models, first, conditions for sustained oscillations will be established. Then

it will be shown that all the conditions of Hopf bifurcation are satisfied and hence either a stable limit cycle or an unstable limit cycle or a nonlinear center exists in the neighborhood of bifurcation line/curve. Limit cycles will be located by numerical experiments for different values of feedback coefficients.

In the last chapter, results obtained will be interpreted and further investigations required will be pointed out.

CHAPTER II

FORMULATION OF DYNAMICAL SYSTEM FOR FISSION REACTORS

In this chapter, the point reactor model with two energy balances and reactivity feedbacks is formulated. These equations are put in a convenient form by suitable translation and normalization of the dependent and independent variables. All the variables are then non-dimensional.

2.1 THE LUMPED PARAMETER DYNAMICAL MODEL

In section 1.2.2, we obtained Eqs. (1.8) and (1.9) for the point reactor model with one group of delayed neutrons. Writing reactor power, P instead of neutron flux density n, and using power units for C, we can rewrite the equations as

$$\frac{dP}{dt} = \frac{\rho-\beta}{l} P + \lambda C \qquad (2.1)$$

$$\frac{dC}{dt} = \frac{\beta}{l} P - \lambda C \qquad (2.2)$$

We shall now incorporate equations obtained from energy balance to Eqs. (2.1)-(2.2) and include their feedback effects in the first equation.

2.1.1 Energy Balances

The lumped parameter thermal model is based on two energy balances: one for the energy contained in all the fuel elements of

the reactor core, and another for the energy stored in the coolant within the core volume.

The energy balance for the fuel elements is given by

$$C_f \frac{dT_f}{dt} = P - H_f (T_f - T_c)$$

where,

C_f = heat capacity of the fuel elements in the reactor core ($J\ K^{-1}$),

H_f = fuel to coolant heat transfer coefficient ($W\ K^{-1}$),

T_f, T_c = average temperatures of the fuel and the coolant respectively.

Dividing all the terms by C_f, we can write the equation as

$$\frac{dT_f}{dt} = KP - \gamma (T_f - T_c) \qquad (2.3)$$

where, $K = 1/C_f$ and $\gamma = H_f/C_f$

The energy balance for the coolant within the core is given by

$$C_c \frac{dT_c}{dt} = H_f (T_f - T_c) - H_c (T_c - T_i)$$

and $$H_c = 2 \dot{m}_c C_c$$

where,

C_c = heat capacity of the coolant within the reactor core ($J\ K^{-1}$),

$\dot{m}_c$ = coolant mass flow rate,

T_i = coolant inlet temperature,

H_c = twice the product of the mass flow rate and specific heat of the coolant ($W\ K^{-1}$).

Dividing all the terms by C_c, we can rewrite the equation as

$$\frac{dT_c}{dt} = \gamma c\,(T_f - T_c) - s\,(T_c - T_i) \qquad (2.4)$$

where $c = C_f/C_c$, $s = H_c/C_c$ and γ has been defined in Eq. (2.3).

2.1.2 Reactivity Feedback Effects

The reactivity $\rho(t)$ that derives a transient is the net effect of contributions arising from the external disturbances to which the transient is due and the thermal hydraulic feedback effects. Therefore, it can be written as

$$\rho(t) = \rho_i\,(t) + \rho_{fb}\,(t) \qquad (2.5)$$

where $\rho_i(t)$ is the reactivity from external disturbances and $\rho_{fb}(t)$ arises due to thermal hydraulic feedback effects. Assuming linear feedback, we will take

$$\rho_{fb}(t) = \alpha_f\,(T_f - T_f^R) + \alpha_c\,(T_c - T_c^R)$$

where,

α_f, α_c = fuel and coolant temperature coefficients of reactivity, respectively, and

T_f^R, T_c^R = some reference values of T_f, T_c respectively.

For convenience we will choose these reference temperatures as the steady state operating temperatures T_f^o and T_c^o. Then ρ_i turns out to be zero for impulse or step reactivity inputs. Therefore, for linear feedback model Eq. (2.5) takes the form

$$\rho = \alpha_f\,(T_f - T_f^o) + \alpha_c\,(T_c - T_c^o) \qquad (2.6)$$

Here ρ is time independent. In actual reactors ρ is a function of time. The form of the function may be very complex and even unknown.

For higher order feedback, we can write

$$\rho = \alpha_f (T_f - T_f^{\circ}) + \alpha_c (T_c - T_c^{\circ}) + F (T_f - T_f^{\circ}, T_c - T_c^{\circ}) \quad (2.7)$$

where $F (T_f - T_f^{\circ}, T_c - T_c^{\circ})$ is a nonlinear function with no constant or linear terms.

2.2 REDUCTION TO NON-DIMENSIONAL FORM

Equations (2.1) - (2.4) together with Eq. (2.6) represent the lumped parameter dynamic model with linear reactivity feedback. It has two fixed points; the shutdown point $(0, 0, T_i, T_i)$ and the operating point $(P_o, C_o, T_f^{\circ}, T_c^{\circ})$, where P_o and C_o are steady state operating values of P and C, respectively, of the system after disturbance. Now, we shall non-dimensionalize these equations with appropriate normalization and translation.

2.2.1 Normalization of State Variables

In Choosing normalization and translation of state variables following points were kept in mind:

(1) Resulting transformed state variables should be dimensionless,

(2) The normalization should not be done with respect to parameters passing through zero,

(3) The number of parameters in the final equations should be minimum,

(4) The operating point should be brought to origin,

(5) the shutdown point should be independent of the parameters, if possible,

(6) The transformed variables and the parameters should allow some direct physical interpretation, as for as possible.

The normalization and translation used are

$$\mathbb{P} = \frac{P - P_o}{P_o} \tag{2.8}$$

$$\mathbb{C} = \frac{C - C_o}{C_o} \tag{2.9}$$

$$\mathbb{T}_f = \frac{T_f - T_f^{\,o}}{T_f^{\,o} - T_i} \tag{2.10}$$

$$\mathbb{T}_c = \frac{T_c - T_c^{\,o}}{T_f^{\,o} - T_i} \tag{2.11}$$

2.2.2 Time Normalization

In deciding time normalization, the criterions were:

(1) A dimensionless time should be used,

(2) Number of parameters should be minimum in the equations,

(3) The orders of magnitude of the final parameters should be reasonable,

(4) The time scale should not become too inflated, so as to require too large integration steps.

Time normalization used was:

$$\tau = \lambda t \tag{2.12}$$

2.2.3 The Dimensionless Equations

Incorporation of these transformations in Eqs. (2.1) to (2.4) and Eqs. (2.6) we obtain the following system of equations:

$$\frac{d\mathbb{P}}{d\tau} = -b\mathbb{P} + b\mathbb{C} + a_f \mathbb{T}_f + a_c \mathbb{T}_c + a_f \mathbb{P}\mathbb{T}_f + a_c \mathbb{P}\mathbb{T}_c \qquad (2.13)$$

$$\frac{d\mathbb{C}}{d\tau} = \mathbb{P} - \mathbb{C} \qquad (2.14)$$

$$\frac{d\mathbb{T}_f}{d\tau} = p(1-\theta)\mathbb{P} - p\,\mathbb{T}_f + p\,\theta\,\mathbb{T}_c \qquad (2.15)$$

$$\frac{d\mathbb{T}_c}{d\tau} = \frac{pc}{\theta}(\mathbb{T}_f - \mathbb{T}_c) \qquad (2.16)$$

A total of six parameters have appeared in the system of equations. They are

$$b = \beta/\lambda l, \quad a_f = \alpha_f (T_f^{\circ} - T_i)/\lambda l,$$

$$a_c = \alpha_c (T_c^{\circ} - T_i)/\lambda l, \quad p = H_f/C_f\lambda,$$

$$\theta = \frac{(T_c^{\circ} - T_i)}{(T_f^{\circ} - T_i)}, \quad c = C_f/C_c = 1/K\,C_c$$

All these parameters as well as state variables are dimensionless.

In terms of dimensionless $\mathbb{P}$, $\mathbb{C}$, $\mathbb{T}_f$ and $\mathbb{T}_c$, the reactor operating or steady state point is (0, 0, 0, 0) and the shutdown point is (-1, -1, -1, -1). This suggests that the values of state variables below -1 would be unrealistic.

2.3 TWO-TEMPERATURE FEEDBACK MODEL WITH DELAYED NEUTRONS

The model used in the previous section was based on two-temperature feedback via two energy balances. Delayed

neutrons were included and feedback assumed was linear.

This model already given in previous section will be reproduced once again.

$$\frac{d\mathbb{P}}{d\tau} = -b\mathbb{P} + b\mathbb{C} + a_f \mathbb{T}_f + a_c \mathbb{T}_c + a_f \mathbb{P}\mathbb{T}_f + a_c\mathbb{P}\mathbb{T}_c \qquad (2.17)$$

$$\frac{d\mathbb{C}}{d\tau} = \mathbb{P} - \mathbb{C} \qquad (2.18)$$

$$\frac{d\mathbb{T}_f}{d\tau} = p(1 - \theta)\,\mathbb{P} - p\,\mathbb{T}_f + p\,\theta\,\mathbb{T}_c \qquad (2.19)$$

$$\frac{d\mathbb{T}_c}{d\tau} = \frac{pc}{\theta}\,(\mathbb{T}_f - \mathbb{T}_c) \qquad (2.20)$$

This system of four coupled nonlinear differential equations continuous dynamical system with six parameters and having at least two fixed points (0, 0, 0, 0) and (-1, -1, -1, -1). The fixed points are found by putting derivatives in Eqs. (2.1) - (2.4) equal to zero.

2.4 TWO-TEMPERATURE FEEDBACK MODEL WITHOUT DELAYED NEUTRONS

If we neglect the delayed neutrons, the parameter b is zero, and Eq. (2.18) is no longer required. Thus the governing system of Eqs. (2.17) - (2.20) become

$$\frac{d\mathbb{P}}{d\tau} = a_f \mathbb{T}_f + a_c \mathbb{T}_c + a_f \mathbb{P}\mathbb{T}_f + a_c\mathbb{P}\mathbb{T}_c \qquad (2.21)$$

$$\frac{d\mathbb{T}_f}{d\tau} = p(1 - \theta)\,\mathbb{P} - p\,\mathbb{T}_f + p\,\theta\,\mathbb{T}_c \qquad (2.22)$$

$$\frac{d\mathbb{T}_c}{d\tau} = \frac{pc}{\theta}\,(\mathbb{T}_f - \mathbb{T}_c) \qquad (2.23)$$

This set of differential equations forms a three-dimensional continuous dynamical system with five parameters having the two obvious fixed points (0, 0, 0) and (-1, -1, -1).

CHAPTER III

INVESTIGATION OF THE DYNAMICAL SYSTEMS

In the last chapter dynamical systems of three and four dimensions were obtained. Now the behavior of these systems will be investigated by analytic as well as numerical methods. The main objective will be to locate limit cycles if any. The main concept of analysis, which is essentially Hopf bifurcation theorem, has been taken from Refs. [15, 20, 18].

3.1 TWO TEMPERATURE FEEDBACK MODEL WITHOUT DELAYED NEUTRONS

This is a three dimensional system of equations.

$$\frac{d\mathbb{P}}{d\tau} = a_f \mathbb{T}_f + a_c \mathbb{T}_c + a_f \mathbb{P}\mathbb{T}_f + a_c \mathbb{P}\mathbb{T}_c \tag{3.1}$$

$$\frac{d\mathbb{T}_f}{d\tau} = p(1 - \theta)\,\mathbb{P} - p\,\mathbb{T}_f + p\,\theta\,\mathbb{T}_c \tag{3.2}$$

$$\frac{d\mathbb{T}_c}{d\tau} = \frac{pc}{\theta}\,(\mathbb{T}_f - \mathbb{T}_c) \tag{3.3}$$

It has two fixed points (0, 0, 0) and (-1, -1, -1). These are the operating and the shutdown points, respectively.

Local Behavior at the Operating Point (0, 0, 0)

The Jacobian matrix at the operating point is

$$\begin{bmatrix} 0 & a_f & a_c \\ p(1-\theta) & -p & p\theta \\ 0 & pc/\theta & -pc/\theta \end{bmatrix}$$

Characteristic equation of this matrix is

$$\lambda^3 + p(1+c/\theta)\,\lambda^2 + (pc/\theta - a_f)\,p\,(1-\theta)\,\lambda - (a_f + a_c)\,p(1-\theta)\,pc/\theta = 0 \qquad (3.4)$$

We write (3.4) as

$$\lambda^3 + e\lambda^2 + f\lambda + g = 0 \qquad (3.5)$$

where,

$$e = p\,(1 + c/\theta), \quad f = (pc/\theta - a_f)\,p\,(1-\theta),$$

$$g = -(a_f + a_c)\,p(1-\theta)\,pc/\theta \qquad (3.6)$$

Since sustained oscillation is a precondition for the existence of limit cycles, we shall examine the conditions of asymptotic stability or sustained oscillations. This can be done with the help of Routh - Hurwitz criterion [31].

3.1.1 Routh-Hurwitz Criterion

The characteristic equation is

$$\lambda^3 + e\lambda^2 + f\lambda + g = 0$$

Routh's Tabulation

λ^3	1	f
λ^2	e	g
λ	ef-g/e	
λ^0	g	

(3.7)

From Routh's table it is obvious that all the roots of Eq. (3.5) will have negative real parts iff the following conditions are fulfilled simultaneously:

$$e > 0, \quad ef-g/e > 0, \quad g > 0$$

Sustained oscillations occur when there exists atleast one pair of conjugate imaginary roots; the remaining root has to be negative real. We get information about sustained oscillations by looking at the Routh table given above and by the following analysis.

When $ef-g = 0$, the λ - row in (3.7) consists only of zeros, so the entry in λ^0 - row will become indeterminate. When a row of zeros occurs, there exists an even polynomial divisor (also called 'auxiliary polynomial') of the original polynomial. The coefficients of this divisor polynomial are given by the previous non-zero row of the array. In our case the coefficients in λ^2 - row are e and g, so the divisor is:

$$e\lambda^2 + g \tag{3.8}$$

This divisor represents a conjugate pair of imaginary roots iff e and g have the same sign.

When Routh- Hurwitz array terminates prematurely. We complete the table by replacing the row of zeros with the coefficients of the derivative with respect to λ of the divisor:

$$\frac{d}{d\lambda} \left[e\lambda^2 + g\right] \tag{3.9}$$

The resulting Routh tabulation takes the form

λ^3	1	f
λ^2	e	g
λ	2e	
λ^0	g	

(3.10)

From the above analysis and the Routh tabulation given in (3.10), it follows that the sustained oscillations occur when the following conditions are fulfilled simultaneously.

$$e > 0, \quad ef - g = 0, \quad g > 0 \tag{3.11}$$

3.1.2 Routh-Hurwitz Criterion Applied to the Present System

From Eq. (3.6), we see that e is always greater than zero, since for any reactor p, c, θ all are positive definite. Also θ is always less than unity which means g > 0 is equivalent to

$$-(a_f + a_c) > 0$$

or

$$(a_f + a_c) < 0 \tag{3.12}$$

Substituting expressions for e, f and g in ef-g = 0 and simplification leads to

$$\frac{\theta}{c} a_f - a_c = p\,(1 + \frac{c}{\theta}) \tag{3.13}$$

Eqs. (3.12) and (3.13) are plotted in Fig. 2 for parameter values p = 2, C = 1/2, θ = 1/20 which represents a typical thermal reactor.

3.1.3 Hopf Bifurcation at Operating Point

Eqs. (3.12) and (3.13) are necessary and sufficient conditions for the roots of Eq. (3.5) having a pair of pure imaginary roots. Now treating p, c, θ, a_c to be fixed it is not difficult to show that the complex conjugate eigenvalues cross the imaginary axis from left to right and the real eigenvalue remains in the left-half plane as a_f increases beyond a_f^*, where a_f^* represents a point on the bifurcation line satisfying Eqs. (3.12) and (3.13). To show this we start by assuming that two of

the eigenvalues are a complex conjugate pair, $R \pm iI$, and the third of course is necessarily real, say, S. Then, by writing the characteristic equation, it is straight forward to see that the following must hold [24]:

$$2R + S = -p(1 + c/\theta)$$

$$R^2 + I^2 + 2RS = p(1-\theta)(pc/\theta - a_f)$$

$$(R^2 + I^2)S = p^2(a_f + a_c)(1 - \theta)c/\theta \qquad (3.14)$$

In order to see the movement of the eigenvalues across the imaginary axis, we differentiate Eqs. (3.14) with respect to a parameter, say a_f, considering the components R, I and S of the roots as variables, and keeping all other parameters fixed. The resulting equations can be solved for the derivative of R, and considering the fact that $c > 0$, $p > 0$, and $1 > \theta > 0$, it is not difficult to show [24] that

$$\left.\frac{dR}{da_f}\right|_{R=0} > 0 \qquad (3.15)$$

Therefore from Hopf bifurcation theorem it follows that there exists either a stable limit cycle for $a_f > a_f^*$ or an unstable limit cycle for $a_f < a_f^*$ or a nonlinear center at $a_f = a_f^*$ and all the three cases are mutually exclusive.

Numerical Experiments

Numerical experiments were performed for $p = 2$, $c = 0.5$, $\theta = 0.05$ for three pairs of values of a_f and a_c. Pandey [24] reports stable limit cycles for negative a_c and positive a_f. We concentrated on the case where both a_c and a_f are negative since

from reactor safety point of view both a_c and a_f negative are important.

Limit cycles have been plotted in figures, as indicated in the Table 1. A point within the cycle spirals out tending towards limit cycle. A point from outside spirals inwards towards the limit cycle.

Results of numerical experiments are summarized in the Table 1.

3.2 TWO TEMPERATURE FEEDBACK MODEL WITH DELAYED NEUTRONS

This is a four-dimensional system of equations

$$\begin{aligned}
\frac{d\mathbb{P}}{d\tau} &= -b\mathbb{P} + b\mathbb{C} + a_f \mathbb{T}_f + a_c \mathbb{T}_c + a_f \mathbb{P}\mathbb{T}_f + a_c \mathbb{P}\mathbb{T}_c \\
\frac{d\mathbb{C}}{d\tau} &= \mathbb{P} - \mathbb{C} \\
\frac{d\mathbb{T}_f}{d\tau} &= p(1-\theta)\,\mathbb{P} - p\,\mathbb{T}_f + p\,\theta\,\mathbb{T}_c \\
\frac{d\mathbb{T}_c}{d\tau} &= \frac{pc}{\theta}\,(\mathbb{T}_f - \mathbb{T}_c)
\end{aligned} \tag{3.16}$$

having two fixed points (0, 0, 0, 0) and (-1, -1, -1, -1).

Local Behavior of the Operating Point (0, 0, 0, 0)

The Jacobian matrix at the operating point is

$$\begin{bmatrix}
-b & b & a_f & a_c \\
1 & -1 & 0 & 0 \\
p(1-\theta) & 0 & -p & p\theta \\
0 & 0 & pc/\theta & -pc/\theta
\end{bmatrix} \tag{3.17}$$

Characteristic equation of this matrix is

$$\lambda^4 + e\lambda^3 + f\lambda^2 + g\lambda + h = 0 \qquad (3.18)$$

where,

$$e = \left(\frac{pc}{\theta} + p + b + 1\right)$$

$$f = p(1-\theta)\left(\frac{pc}{\theta} - a_f\right) + (b+1)\,p\,\left(1 + \frac{c}{\theta}\right)$$

$$g = \frac{pc}{\theta}(b+1)\,p(1-\theta) - a_f\left(\frac{pc}{\theta} + 1\right)p\,(1-\theta) - a_c\,\frac{pc}{\theta}\,p(1-\theta)$$

$$h = -\frac{pc}{\theta}\,p(1-\theta)\,(a_f + a_c) \qquad (3.19)$$

For sustained oscillations like three-dimensional case, Eq. (3.18) should also have at least a pair of pure imaginary roots. The other pair can either be complex conjugates with negative real parts or negative reals. The necessary and sufficient conditions for a pair of roots to be complex conjugate can be determined using Routh-Hurwitz criterion.

3.2.1 Routh-Hurwitz Criterion

The characteristic equation is

$$\lambda^4 + e\lambda^3 + f\lambda^2 + g\lambda + h = 0$$

Routh's Tabulation

λ^4	1	f	h
λ^3	e	g	
λ^2	ef-g/e	h	
λ	$g-e^2h/(ef-g)$		
λ^0	h		

From Routh's table it is obvious that all the roots of Eq. (3.18) will have negative real parts iff the following conditions are fulfilled simultaneously.

$$e > 0; \quad ef-g > g; \quad g - \frac{e^2 h}{ef-g} > 0 \text{ and } h > 0 \tag{3.20}$$

We can get information about sustained oscillations by looking at the Routh table given above and the following analysis.

When $g(ef-g) - e^2 h = 0$, the λ - row consists solely of zeros, so the entry in the λ^0 - row would be indeterminate. When a row of zeros occurs, there exists an even polynomial divisor (also called 'auxiliary polynomial') of the original polynomial. The coefficients of this divisor polynomial are given by the previous nonzero row of the array. In the present problem, the coefficients in the λ^2-row are ef-g/e and h, so the divisor is:

$$\frac{ef - g}{e} \lambda^2 + h \tag{3.21}$$

This divisor represents a conjugate pair of imaginary roots iff (ef-g) and h have the same sign.

To compute the Routh-Hurwitz array after a permature termination, we replace the row of zeros with the coefficients of the derivatives with respect to λ of the divisor:

$$\frac{d}{d\lambda} \left[\frac{ef-g}{e} \lambda^2 + h \right] \tag{3.22}$$

The resultant Routh tabulation is as follows:

λ^4	1	f	h
λ^3	e	g	
λ^2	ef-g/e	h	
λ	2(ef-g)/e		
λ^0	h		

(3.23)

From the above analysis and the Routh tabulation given in (3.23), it follows that the sustained oscillations occur when the following conditions are fulfilled simultaneously:

$$g > 0, \quad h > 0 \quad \text{and} \quad g(ef-g) - e^2h = 0 \tag{3.24}$$

Using the reactor parameters p = 2.0, c = 0.5, θ = 0.05, b = 88.0, from Eqs. (3.19) and (3.20) we get

$$3382 - 39.9\, a_f - 38\, a_c > 0 \tag{3.25}$$

$$-38\,(a_f + a_c) > 0 \tag{3.26}$$

$$a_f^2 - 0.21\, a_c^2 + 0.73\, a_f a_c - 1620\, a_f - 1440.5\, a_c + 135985.8 = 0 \tag{3.27}$$

Eq. (3.27) represents a two-sheet hyperbola which has been plotted in Fig. 8. Imposition of conditions (3.25) and (3.26) on it gives the bifurcation curve. Three possible zones are shown in the figure.

For all possible values of a_f and a_c on bifurcation curve, characteristic Eq. (3.18) has a pair of pure imaginary roots. The other two turn out to be pure real negatives. Now in order that all the conditions of Hopf bifurcation theorem is satisfied what remains to be shown is that complex conjugate eigenvalues cross the imaginary axis from left to right as one of the parameters (a_f or a_c) is increased. To show this analytically involves unnecessary algebra. We can show the crossing numerically. For this we evaluate the roots of characteristic equation at a_f^*, $a_f > a_f^*$ and $a_f < a_f^*$ for a number of points and observe that indeed

crossing is occurring. Crossing of eigenvalues as well as movement of negative real roots are shown in Fig. 7.

Numerical results for a few points are given in the Appendix C.

Therefore, from Hopf bifurcation theorem it follows that there exists a limit cycle in a sufficiently small neighbourhood of bifurcation curve. Limit cycle is stable for $a_f > a_f^*$, unstable for $a_f < a_f^*$ and for $a_f = a_f^*$ there is a nonlinear center and all these three cases are mutually exclusive.

Numerical Experiment

Numerical experiments were done for p = 2, c = 0.5, θ = 0.05, b = 88.0 for five pairs of values of a_f and a_c. Limit cycle obtained was stable in all the five cases.

Results of numerical experiments are summarized in Table 2.

Starting from any point in four dimension leads us asymptotically to the stable limit cycles provided we are not too far from the operating point. If we start from a point within the limit cycle as seen on $\mathbb{T}_f$ - $\mathbb{P}$ projection it tends to the limit cycle. Similarly, if we start from outside as seen in $\mathbb{T}_f$ - $\mathbb{P}$ projection, the path may appear to cut the limit cycle but in reality, that is in four dimension, this is not the case. Otherwise, our system $\frac{d\underline{X}}{d\tau} = \underline{f}(\underline{X})$ will take more than two values for the same values of the coordinates. Thus violating the single valuedness of the derivative.

The results obtained in case of four-dimensional case shows some remarkable differences from the three-dimensional case. Firstly in the case where delayed neutrons have been taken into consideration has smaller time period than when delayed neutrons are neglected. Also amplitude becomes too large in case of model with delayed neutrons.

3.3 NUMERICAL METHOD EMPLOYED FOR SOLVING THE SYSTEM OF DIFFERENTIAL EQUATIONS

The Jacobian matrix for our dynamical system have the eigenvalues which are very widely separated. This implies, as pointed out in introduction that some transients may last for a few seconds and some may last for several hours. Such systems cannot be solved quite accurately by methods such as Runge-Kutta or Adam-Bashforth which is based on multiple predictor-corrector method.

To solve our system, the subroutine D02EBF from NAG library was employed on DEC-1090. Subroutine D02EBF is based on Gear method. In case of three-dimensional system tolerance used was 0.1E - 05 and in case of four-dimensional system it was 0.1E-07 which gives quite accurate solution as was seen by using tolerance smaller than these.

The search for limit cycle was initiated by using a point close to operating point as initial condition. The last step of the iteration was again used as the initial condition and it was repeated till limit cycle was finally located. Starting from any other point asymptotically converges to the limit cycle. To have better accuracy the step size were taken small. In case of

three-dimensional system step size was 0.01 and in four-dimensional case, step size was 0.001.

NAG subroutine C02AEF was used to show numerically the crossing of eigenvalues in four dimensional system. The subroutine C02AEF calculates all the roots of a real polynomial of order upto hundred using the method of Grants and Hitchins.

The next chapter briefly summarizes the findings of this work.

CHAPTER IV

CONCLUSIONS

4.1 SUMMARY OF INVESTIGATION

We started with two temperature model, i.e. two energy balances together with reactor kinetics equations to get a set of differential equations. Further these differential equations were coupled taking feedback proportional to fuel and coolant temperatures. This feedback essentially arises because of thermal-hydraulic effects. The resulting set of equations were non-dimensionalized (since it is convenient to analyze non-dimensional equations) keeping into mind that number of parameters is as small as possible, new state variables have some direct physical interpretations.

After this, models with and without delayed neutrons were analyzed for Hopf bifurcation. Using Routh-Hurwitz criterion, the necessary and sufficient conditions for the characteristic equations to have a pair of pure imaginary roots were obtained. Parameters a_f and a_c were treated as variables where as other parameters as fixed. Further, it was shown that with respect to parameter a_f crossing of imaginary eigenvalues is occurring while others remain on right hand plane. For model without delayed neutrons crossing was shown analytically. For model with delayed neutrons crossing was shown numerically. Then using Hopf

bifurcation theorem the existence of limit cycles were predicted. Numerical experiments were done to locate these cycles and were found to be stable in both the cases that is with and without delayed neutrons.

4.2 FURTHER INVESTIGATIONS REQUIRED

While studying the dynamics of a thermal reactor we have concentrated on the core of the reactor. We have neglected the interactions of controls and plant with the core. Further studies in this direction can be done taking these interactions into account.

4.3 EPILOGUE

In the present work two models, one without delayed neutrons and the other with delayed neutrons, were studied using the concept of Hopf bifurcation. For the conditions of the theorem to be satisfied, and therefore the limit cycles to exist, it was found that coolant feedback coefficient has to be negative. Fuel feedback coefficient may be negative as well as positive. For both the models stable limit cycles were computed numerically.

APPENDIX A

1.1 STIFF SYSTEMS

"Stiffness" is a property of a mathemtical problem (not of the numerical solution method). Let us consider the system of differential equations which are first order, nonlinear and ordinary.

$$\frac{dy}{dx} - f(x, y) = 0 \quad y(a) = y_o, \quad x \in [a, b],$$

The above initial value problem is said to be <u>stiff</u> in an interval $I \subset [a, b]$ if, for $x \in I$,

(1) $Re(\lambda_i) < 0 \quad (i = 1, 2, \ldots s)$; and

(2) $$S(x) = \max_{i=1,s} Re(-\lambda_i) / \min_{i=1,s} Re(-\lambda_i) >> 0$$

where the λ_i are the eigen values of $\partial f/\partial y$ evaluated on the solution $y(x)$ at x.

The ratio $S(x)$ may be termed the (local) "stiffness ratio" of the problem. Problems may be considered to be marginally stiff if $S(x)$ is $O(10)$, while stiffness ratio upto $O(10^6)$ are not uncommon in practical problems arising in such fields as chemical kinetics, process control and reactor kinetics.

A stiff problem is often referred to in the literature as a problem wilth "widely differing time constants" or as a system with a large "Lipscitz constant".

1.2 GEAR'S METHOD OF SOLVING STIFF SYSTEMS

The NAG subroutine D02EBF used for calculation is based on Gear's method. The subroutine D02EBF is suited both for non-stiff as well as stiff systems. If the required system of differential equations is non-stiff, Adams-Bashforth p-th order predictor - corrector technique is used. The order p of predictor - corrector is decided by the subroutine itself.

The system of N-differential equations are:

$$\frac{dy}{dt} = f(y, t) \qquad (A.1)$$

In case of Adam-Bashforth method predictor equation of order p is given by

$$y_{n,o} = y_{n-1} + \beta_1 h y'_{n-1} + \ldots + \beta_p h y'_{n-p} \qquad (A.2)$$

where

$y_i = y(t_i) = ih$ (h being the step size),

$y'_i = f(y_i, t_i)$,

and where the β_i are given, for example, in Henrici [35]. The approximation $y_{n,(o)}$ in this predictor corrector scheme as the first approximation in the Adams-Moulton corrector formula of the p-th order given by

$$y_{n,(m+1)} = y_{n-1} + \beta_0^* h f(y_{n,(m)}, t_n) + \beta_1^* h y'_{n-1} + \ldots + \beta^*_{p-1} h y'_{n-p+1} \qquad (A.3)$$

The coefficients β_i^* can be found in Henrici [35]. If the corrector equation (A.3) is iterated until it converges to y_n (as is guaranteed for small enough step size and smooth function f) the

1.2 GEAR'S METHOD OF SOLVING STIFF SYSTEMS

The NAG subroutine D02EBF used for calculation is based on Gear's method. The subroutine D02EBF is suited both for non-stiff as well as stiff systems. If the required system of differential equations is non-stiff, Adams-Bashforth p-th order predictor - corrector technique is used. The order p of predictor - corrector is decided by the subroutine itself.

The system of N-differential equations are:

$$\frac{dy}{dt} = f(y, t) \qquad (A.1)$$

In case of Adam-Bashforth method predictor equation of order p is given by

$$y_{n,o} = y_{n-1} + \beta_1 h y'_{n-1} + \ldots + \beta_p h y'_{n-p} \qquad (A.2)$$

where

$y_i = y(t_i) = ih$ (h being the step size),

$y'_i = f(y_i, t_i)$,

and where the β_i are given, for example, in Henrici [35]. The approximation $y_{n,(o)}$ in this predictor corrector scheme as the first approximation in the Adams-Moulton corrector formula of the p-th order given by

$$y_{n,(m+1)} = y_{n-1} + \beta_o^* h f(y_{n,(m)}, t_n) + \beta_1^* h y'_{n-1} + \ldots + \beta^*_{p-1} h y'_{n-p+1} \qquad (A.3)$$

The coefficients β_i^* can be found in Henrici [35]. If the corrector equation (A.3) is iterated until it converges to y_n (as is guaranteed for small enough step size and smooth function f) the

truncation error introduced in the n-th step of the integration will be $C^{A}_{p+1}\ h^{p+1}\ y^{(p+1)}\ (t_n) + O\ (h^{p+2})$, where $y^{(k)}$ is the k-th derivative of y. The coefficients C^{A}_{p+1} are given in Henrici [35].

The method for stiff equations is similar. It uses a p-th order predictor formula of the form

$$y_{n,(o)} = \alpha_1\ y_{n-1} + \ldots + \alpha_p\ y_{n-p} + \eta_1\ h\ y'_{n-1} \qquad (A.4)$$

and a corrector

$$y_{n,(m+1)} = \alpha_1^{*}\ y_{n-1} + \ldots + \alpha_p^{*}\ y_{n-p} + \eta_o^{*}\ h\ f(y_{n,(m)},\ t_n) \qquad (A.5)$$

The integration truncation error when (A.4) is iterated to convergence is $C^{S}_{p+1}\ h^{p+1}\ y^{(p+1)}\ (t_n) + O\ (h^{p+2})$, where $C^{S}_{p+1} = \frac{1}{p+1}$. The α_i^{*} and η_o^{*} are given in [38].

APPENDIX B

HOPF BIFURCATION THEOREM

Let us consider a system of ordinary differential equations

$$\frac{dX}{dt} = f(X, \alpha) \tag{B.1}$$

where,

$$X = (x_1, x_2, \ldots, x_n)^T,$$

$$f = (f_1, f_2, \ldots, f_n)^T,$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m). \tag{B.2}$$

Here x_i denote state variables, t is time, and α is generally a vector of system parameters. The system is autonomous, i.e., the time variable does not occur in the right-hand sides explicitly. It shall be assumed that RHS are continuous and continuously differentiable functions of their arguments.

The steady-state (stationary) solution x of system (B.1) satisfies the set of nonlinear equations

$$f_i(x_1, x_2, \ldots, x_n, \alpha) = 0, \quad i = 1,2,\ldots, n \tag{B.3}$$

Let us consider a steady-state solution x of Eqs. (B.3) for $\alpha = \alpha^+$ and a branch of stationary solutions $x(\alpha)$ in the neighborhood of $\alpha = \alpha^+$. The functions f_i are taken to be sufficiently smooth.

Let us assume that all eigen values of the Jacobian matrix J are nonzero and that only two eigenvalues are purely imaginary; let us call them $\lambda(\alpha)$ and $\bar{\lambda}(\alpha)$ in the neighborhood of $\alpha = \alpha^+$. Then,

$$\text{Re } (\lambda(\alpha^+)) = \text{Re } (\bar{\lambda}(\alpha^+)) = 0 \qquad \text{(B.4)}$$

Let us further assume that the real part of $\lambda'(\alpha^+)$ is non zero; that is,

$$\text{Re } (\lambda'(\alpha^+)) \neq 0 \qquad \text{(B.5)}$$

We then conclude that a branch of periodic solutions of (B.1) exists for $\alpha > \alpha^+$, $\alpha < \alpha^+$, or $\alpha = \alpha^+$. In greater detail: then there exist functions $X(t, \varepsilon)$, $T(\varepsilon)$, and $\alpha(\varepsilon)$ defined for all t and for a small ε, such that:

1. $\alpha(\varepsilon)$ and $T(\varepsilon)$ are power series in ε.
2. $T(0) = 2\pi/|\lambda(\alpha^+)|$ and $\alpha(0) = \alpha^+$.
3. $X(t, 0) = \bar{X}$, but $X(t, \varepsilon) \neq \bar{X}(\alpha(\varepsilon))$.
4. $X(t, \varepsilon)$ is a solution of equation.
   $X' = f(X, \alpha(\varepsilon))$, with period $T(\varepsilon)$.
5. $\alpha'(\varepsilon)$ is either non-positive, or non-negative, or identically equal to zero for small ε.

If all eigenvalues of J have negative real parts for $\alpha < \alpha^+$, one of the following alternatives holds:

(a) $\alpha(\varepsilon) > \alpha^+$ for all small $\varepsilon \neq 0$ and each periodic solution $X(t, \varepsilon)$ is stable;

(b) $\alpha(\varepsilon) < \alpha^+$ for all small $\varepsilon \neq 0$ and each periodic solution is unstable.

CENTRAL LIBRARY
107894

APPENDIX C

NUMERICAL STUDY OF THE MOVEMENT OF EIGENVALUES IN THE COMPLEX PLANE IN THE VICINITY OF A SUSPECTED HOPF BIFURCATION IN TWO-TEMPERATURE REACTOR MODEL WITH DELAYED NEUTRONS.

AC=-.856414
AF*=-355.75
AF=-355.75=AF*

I	REZ(I)	IMZ(I)
1	-0.11000E+03	0.00000E+00
2	0.31188E-04	-0.55392E+02
3	0.31188E-04	0.55392E+02
4	-0.99704E+00	0.00000E+00

AF=-350.0 > AF*

1	-0.11007E+03	0.00000E+00
2	0.32433E-01	-0.55358E+02
3	0.32433E-01	0.55358E+02
4	-0.99704E+00	0.00000E+00

AF=-360.0 < AF*

1	-0.10996E+03	0.00000E+00
2	-0.23927E-01	-0.55417E+02
3	-0.23927E-01	0.55417E+02
4	-0.99704E+00	0.00000E+00

```
          AC=-1.00E+04
          AF*=-689.125
          AF=-689.125=AF*

1      -0.11000E+03        0.00000E+00
2      -0.62712E-07       -0.60841E+02
3      -0.62712E-07        0.60841E+02
4      -0.99755E+00        0.00000E+00
          AF=-680.0 > AF*

1      -0.11010E+03        0.00000E+00
2       0.49356E-01       -0.60788E+02
3       0.49356E-01        0.60788E+02
4      -0.99755E+00        0.00000E+00
          AF=-695.0 < AF*

1      -0.10994E+03        0.00000E+00
2      -0.31795E-01       -0.60875E+02
3      -0.31795E-01        0.60875E+02
4      -0.99755E+00        0.00000E+00
          AC=-0.65E+04
          AF*=88.768
          AF=88.768=AF*

1      -0.11000E+03        0.00000E+00
2      -0.11085E-05       -0.47157E+02
3      -0.11085E-05        0.47157E+02
4      -0.99592E+00        0.00000E+00
          AF=89.0 > AF*

1      -0.11001E+03        0.00000E+00
2       0.13824E-02       -0.47156E+02
3       0.13824E-02        0.47156E+02
4      -0.99592E+00        0.00000E+00
          AF=87.0 < AF*

1      -0.10998E+03        0.00000E+00
2      -0.10555E-01       -0.47168E+02
3      -0.10555E-01        0.47168E+02
4      -0.99592E+00        0.00000E+00
```

AC=-1.90E+04
AF*=-466.88
AF=-466.88=AF*

1	-0.11000E+03	0.00000E+00
2	0.90225E-06	-0.57266E+02
3	0.90225E-06	0.57266E+02
4	-0.99723E+00	0.00000E+00

AF=-460.0 > AF*

1	-0.11008E+03	0.00000E+00
2	0.36238E-01	-0.57225E+02
3	0.36238E-01	0.57225E+02
4	-0.99723E+00	0.00000E+00

AF=-170.0 < AF*

1	-0.10997E+03	0.00000E+00
2	-0.17345E-01	-0.57284E+02
3	-0.17345E-01	0.57284E+02
4	-0.99723E+00	0.00000E+00

AC=-0.80E+04
AF*=-244.63
AF=-244.63=AF*

1	-0.11000E+03	0.00000E+00
2	-0.58311E-05	-0.53452E+02
3	-0.58311E-05	0.53452E+02
4	-0.99682E+00	0.00000E+00

AF=-240.0 > AF*

1	-0.11006E+03	0.00000E+00
2	0.26154E-01	-0.53424E+02
3	0.26154E-01	0.53424E+02
4	-0.99682E+00	0.00000E+00

AF=-250.0 < AF*

1	-0.10994E+03	0.00000E+00
2	-0.30707E-01	-0.53485E+02
3	-0.30707E-01	0.53485E+02
4	-0.99682E+00	0.00000E+00

```
                AC=-0.85E+04
                AF*=-578.0034
                AF=-57°.0034=AF*

1       -0.11000E+03        0.00000E+00
2       -0.11455E-05       -0.59080E+02
3       -0.11155E-05        0.59080E+02
4       -0.99740E+00        0.00000E+00
                AF=-575.0  > AF*

1       -0.12216E+03        0.00000E+00
2        0.60785E+01       -0.52416E+02
3        0.60785E+01        0.52416E+02
4       -0.99708E+00        0.00000E+00
                AF=-580.0  < AF*

1       -0.10998E+03        0.00000E+00
2       -0.10951E-01       -0.59092E+02
3       -0.10951E-01        0.59092E+02
4       -0.99740E+00        0.00000E+00
                AC=-0.75E+04
                AF=-135.0  < AF*
                AF=-133.65=AF*

1       -0.11000E+03        0.00000E+00
2       -0.20896E-05       -0.51440E+02
3       -0.20896E-05        0.51440E+02
4       -0.99657E+00        0.00000E+00
                AF=-131.0  > AF*

1       -0.11004E+03        0.00000E+00
2        0.17877E-01       -0.51423E+02
3        0.17877E-01        0.51423E+02
4       -0.99656E+00        0.00000E+00
                AF=-135.0  < AF*

1       -0.10998E+03        0.00000E+00
2       -0.86975E-02       -0.51449E+02
3       -0.86975E-02        0.51449E+02
4       -0.99657E+00        0.00000E+00
```

```
AC=-0.70E+04
AF*=-22.37
AF=-22.37=AF*

1     -0.11000E+03        0.00000E+00
2     -0.55041E-07       -0.49345E+02
3     -0.55041E-07        0.49345E+02
4     -0.99627E+00        0.00000E+00
      AF=-20.0 > AF*

1     -0.11003E+03        0.00000E+00
2      0.13925E-01       -0.49330E+02
3      0.13925E-01        0.49330E+02
4     -0.99627E+00        0.00000E+00
      AF=-24.0 < AF*

1     -0.10999E+03        0.00000E+00
2     -0.96012E-02       -0.49355E+02
3     -0.96012E-02        0.49355E+02
4     -0.99627E+00        0.00000E+00

AC=-1.05E+04
AF*=-800.25
AF=-800.25=AF*

1     -0.34678E-05       -0.62552E+02
2     -0.34678E-05        0.62552E+02
3     -0.11000E+03        0.00000E+00
4     -0.99768E+00        0.00000E+00
      AF=-798.0 > AF*

1      0.11987E-01       -0.62539E+02
2      0.11987E-01        0.62539E+02
3     -0.11003E+03        0.00000E+00
4     -0.99768E+00        0.00000E+00
      AF=-804.0 < AF*

1     -0.20050E-01       -0.62574E+02
2     -0.20050E-01        0.62574E+02
3     -0.10996E+03        0.00000E+00
4     -0.99768E+00        0.00000E+00
```

AC=-1.1[illegible]E+04
AF*=-911.37
AF=-911.37=AF*

1	-0.11000E+03	0.00000E+00
2	-0.22309E-05	-0.64217E+02
3	-0.22309E-05	0.64217E+02
4	-0.99780E+00	0.00000E+00

AF=-910.0 > AF*

1	-0.11002E+03	0.00000E+00
2	0.71913E-02	-0.64209E+02
3	0.71913E-02	0.64209E+02
4	-0.99780E+00	0.00000E+00

AF=-912.0 < AF*

1	-0.11000E+03	0.00000E+00
2	-0.33181E-02	-0.64221E+02
3	-0.33181E-02	0.64221E+02
4	-0.99780E+00	0.00000E+00

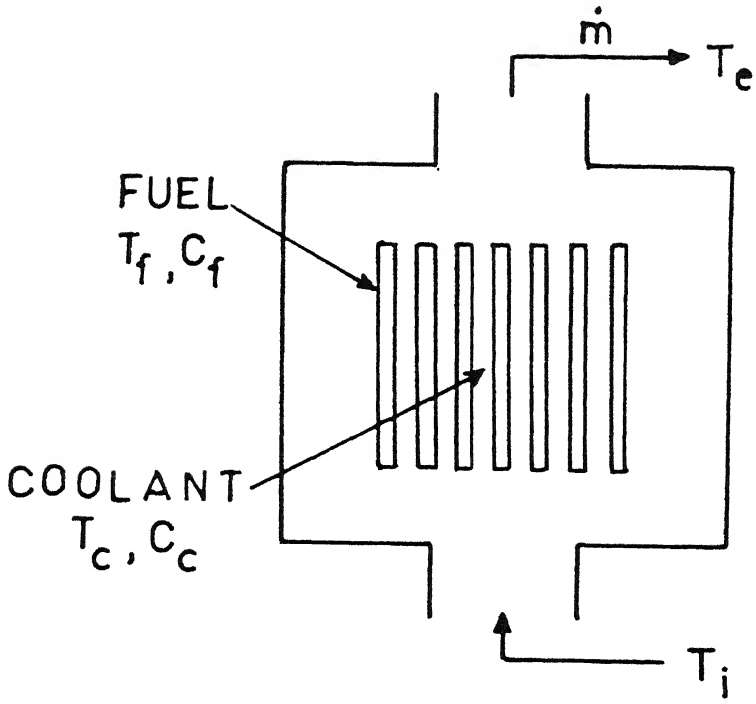
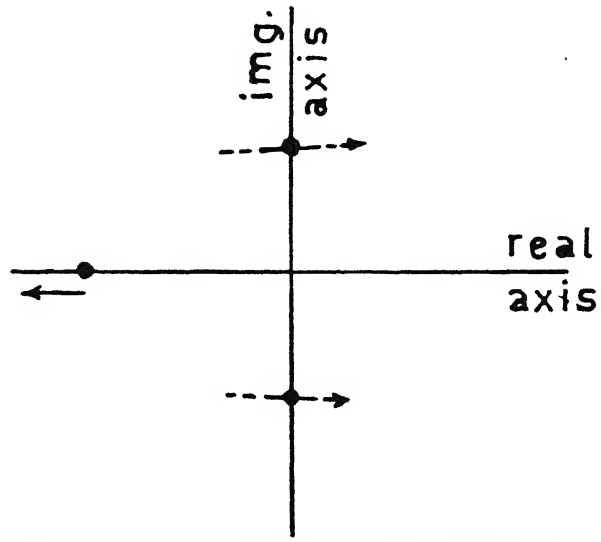


FIG. 1 SCHEMATIC DIAGRAM OF TWO-TEMP. REACTOR MODEL



• location of eigen values at bifurcation (arrows show movement of eigen values as a_f increases)

FIG 2 HOPF BIFURCATION IN TWO - TEMP. MODEL

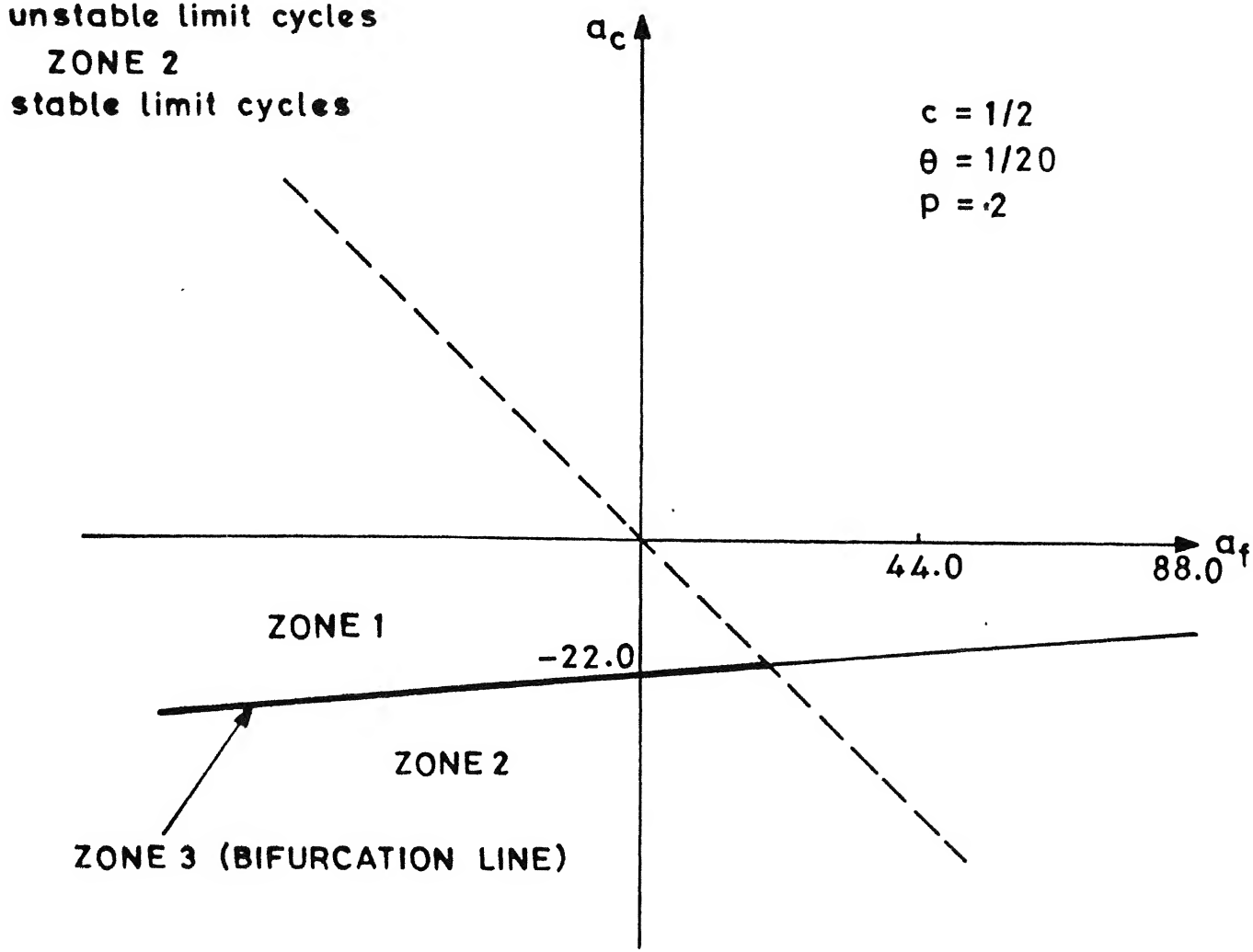


FIG. 3 POSSIBLE ZONES OF PERIODIC BEHAVIOR IN TWO-TEMPERATURE REACTOR-MODEL WITHOUT DELAYED NEUTRONS

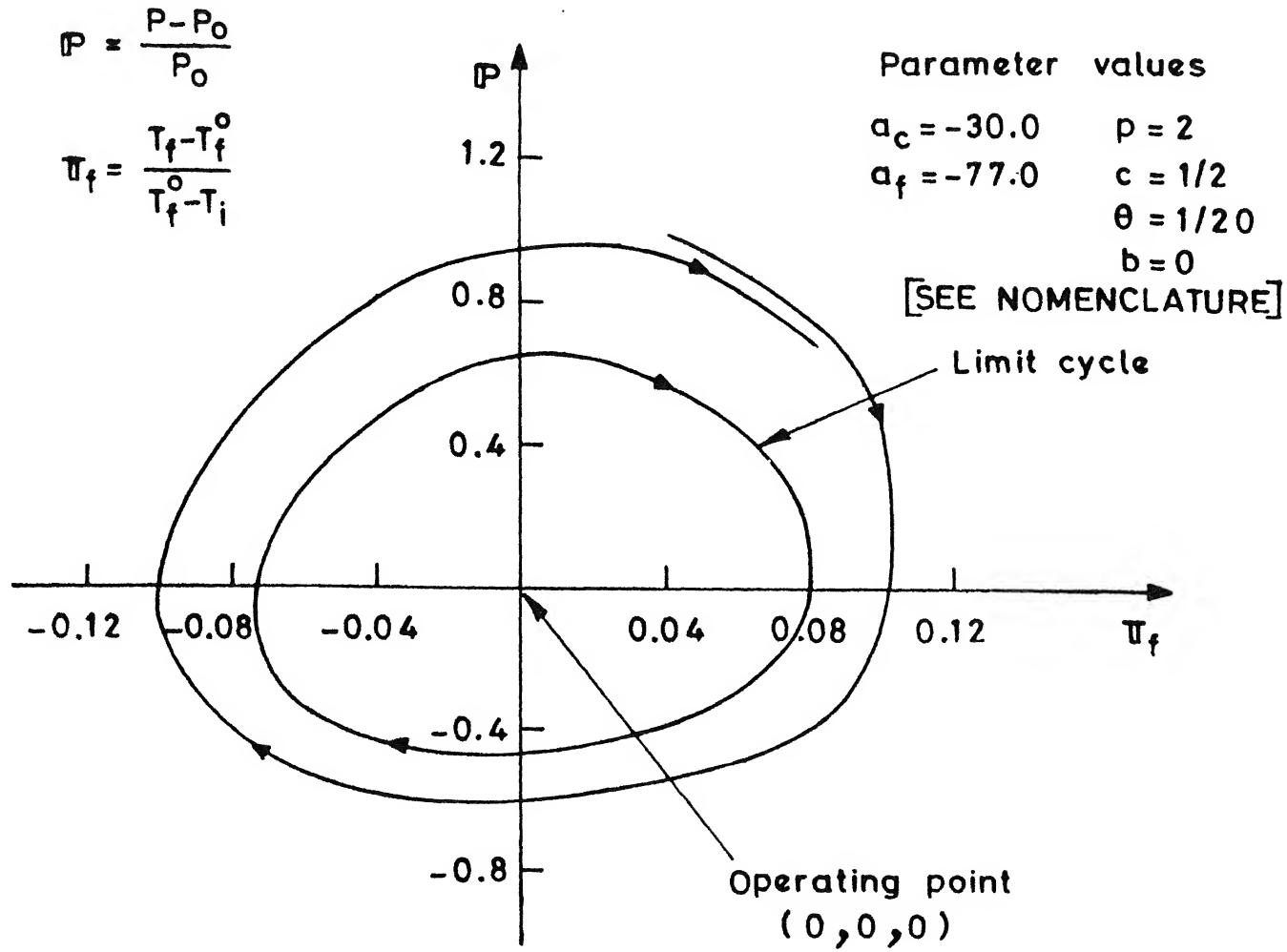


FIG. 4 PROJECTION OF A TRANSIENT ORBIT ON $\mathbb{T}_f$ Vs. $\mathbb{P}$ PLANE WITHOUT DELAYED NEUTRANS (Negative fuel and coolant feedback)

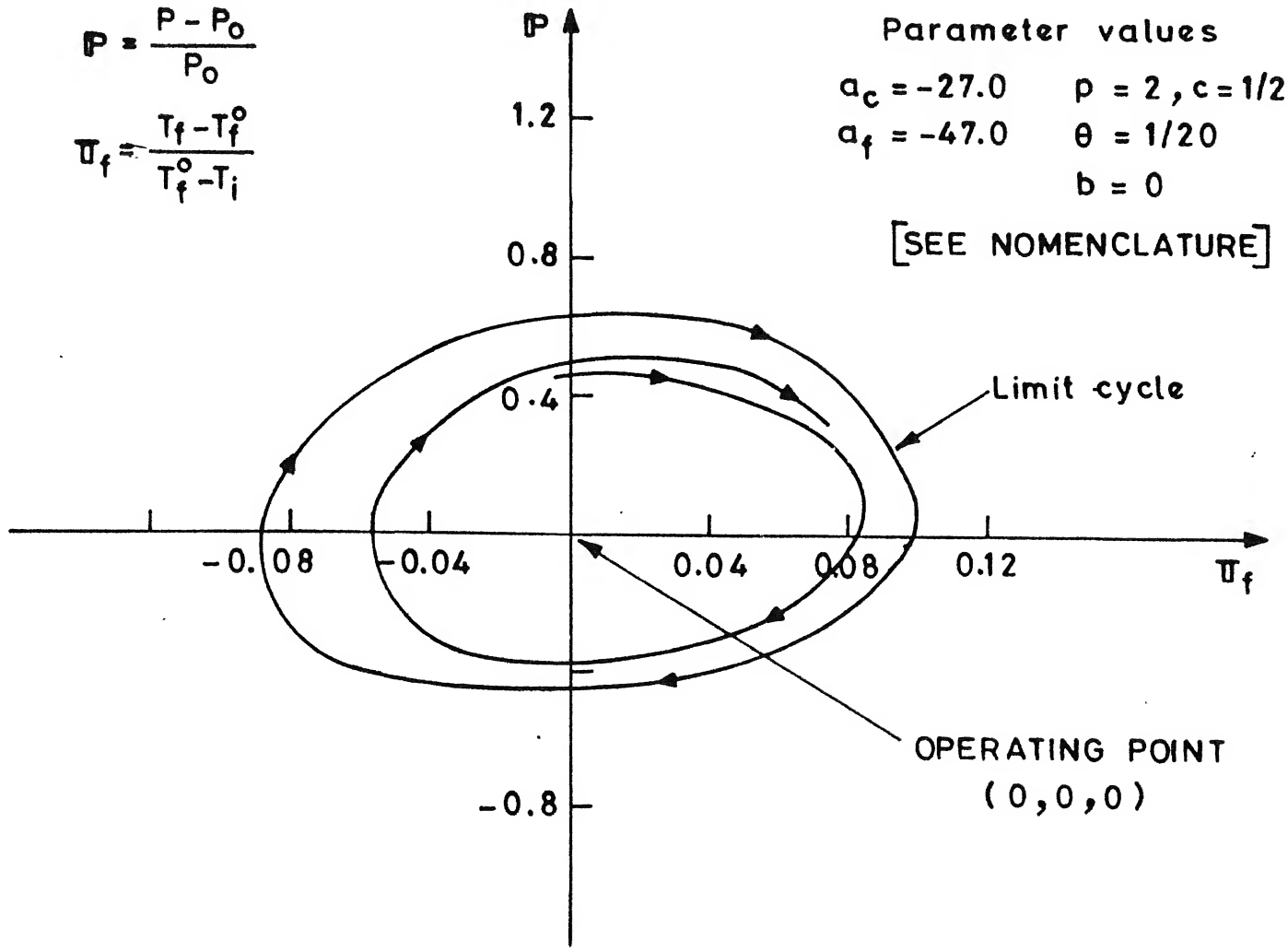


FIG. 5 PROJECTION OF ANOTHER TRANSIENT ORBIT ON π_f Vs. $\mathbb{P}$ PLANE WITHOUT DELAYED NEUTRONS (Different values of a_c and a_f)

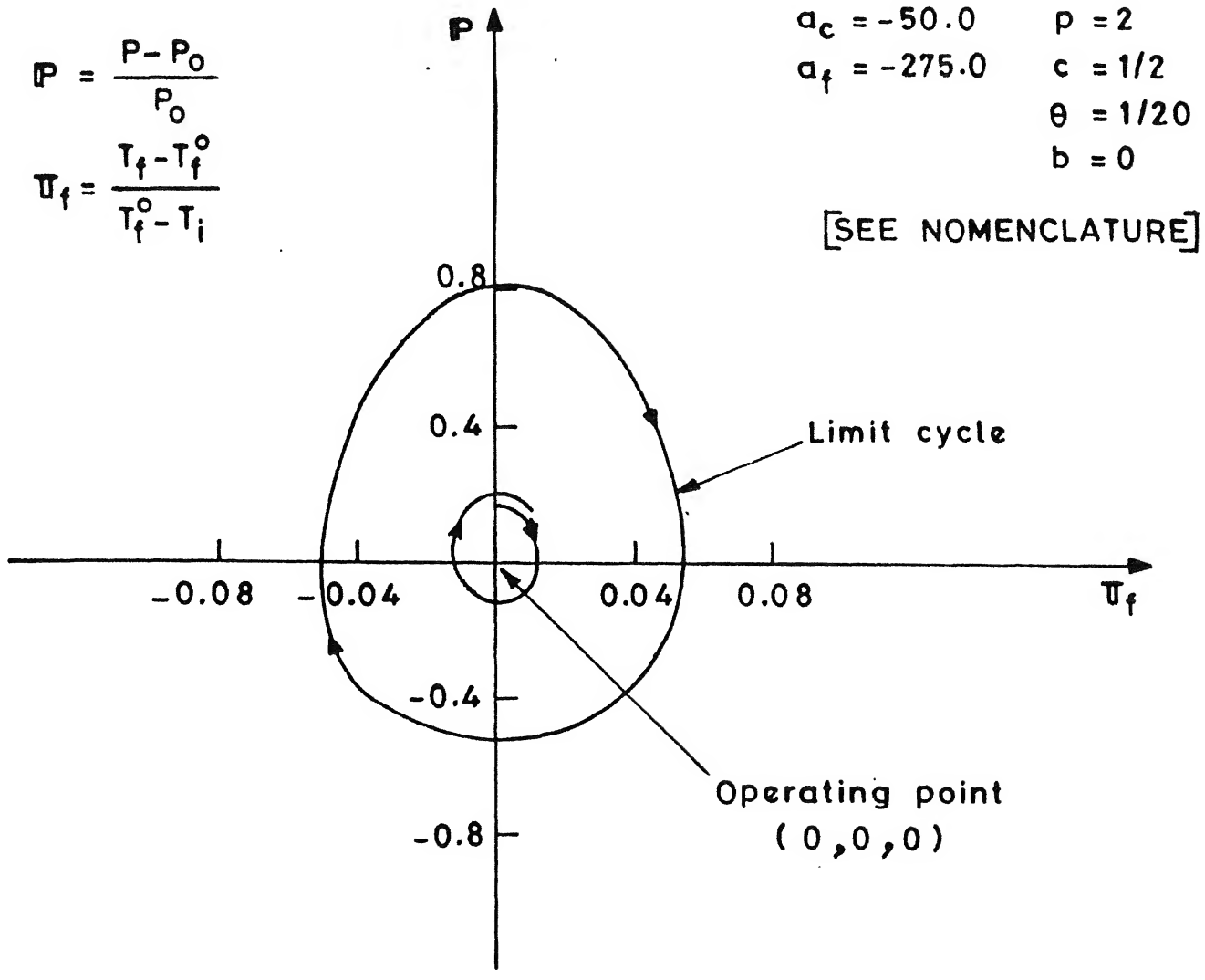


FIG. 6 PROJECTION OF A TRANSIENT ORBIT ON Π_f Vs. $\mathbb{P}$ PLANE
(Different values of a_c and a_f)

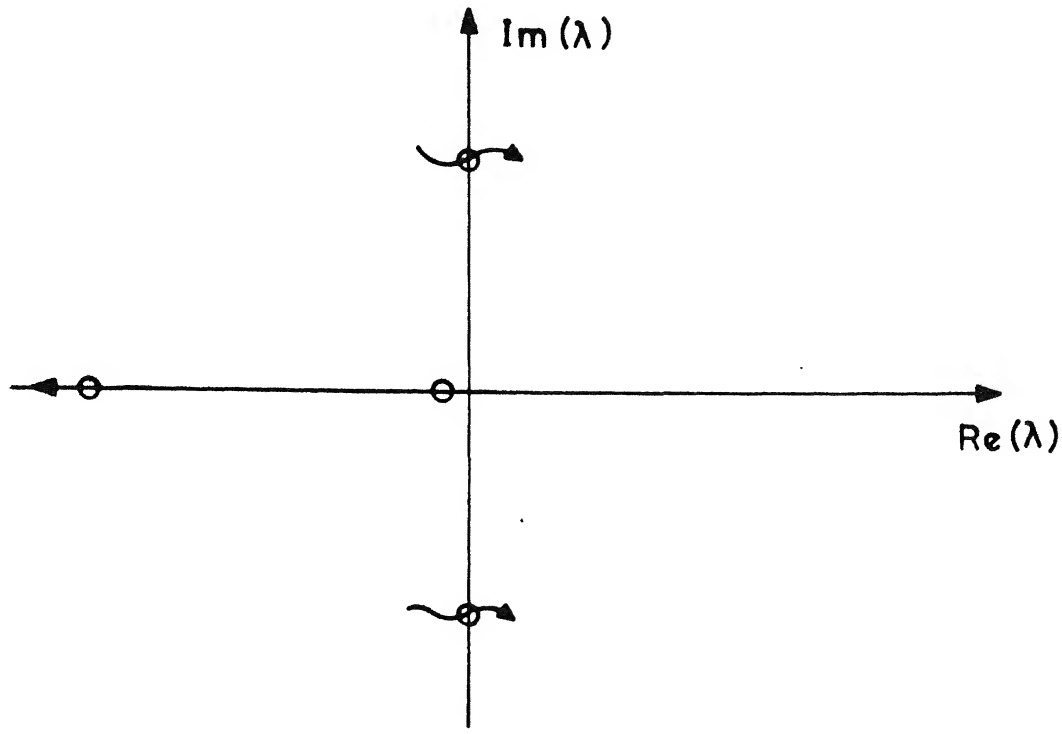


FIG. 7 MOVEMENTS OF EIGENVALUES AS a_f INCREASES BEYOND a_f^* IN TWO TEMPERATURE FEEDBACK MODEL WITH DELAYED NEUTRONS.
(The real eigenvalue close to the origin remains almost stationary in numerical calculation)

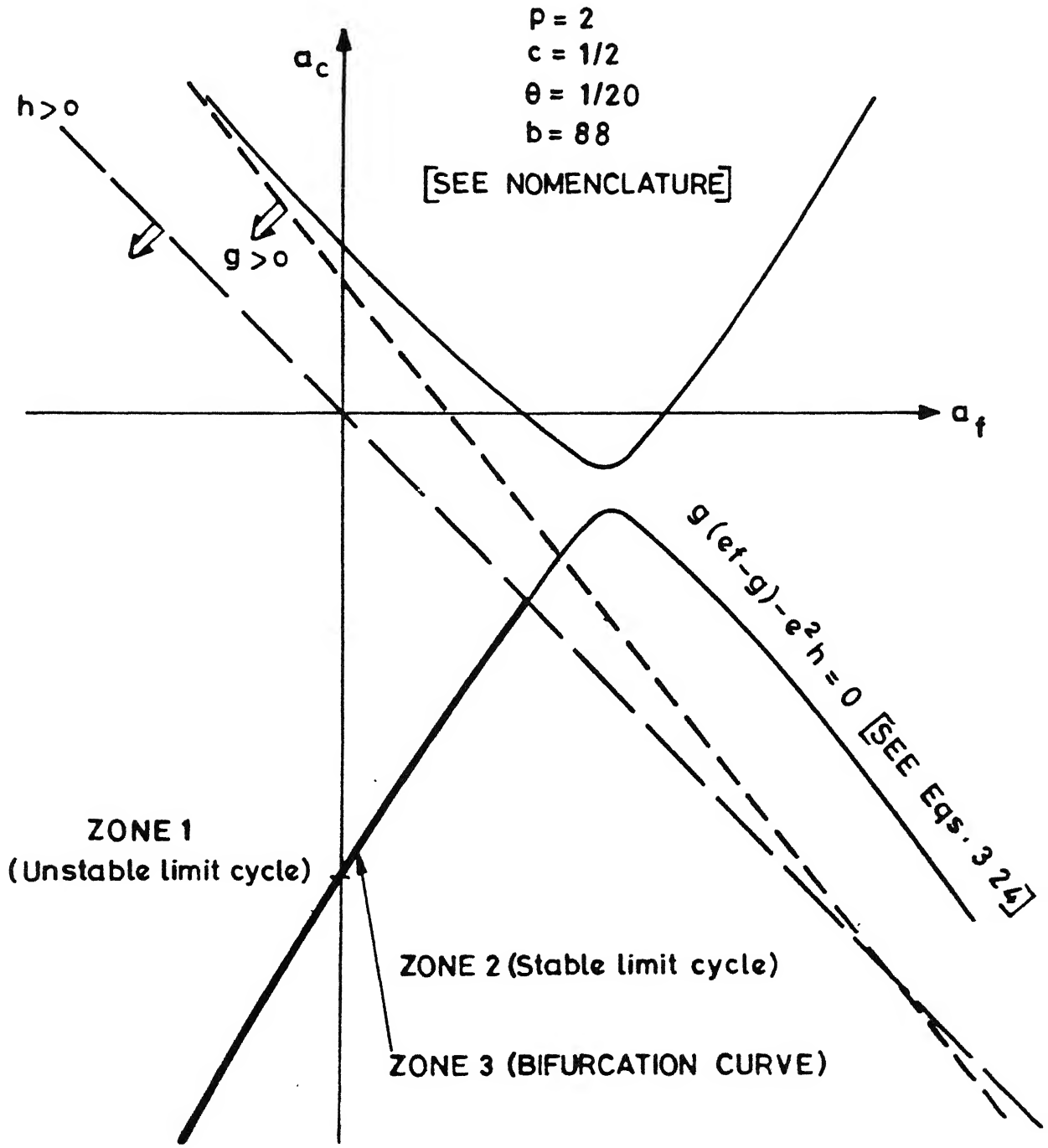


FIG. 8 POSSIBLE ZONES OF PERIODIC BEHAVIOR IN TWO TEMPERATURE REACTOR MODEL WITH DELAYED NEUTRONS.

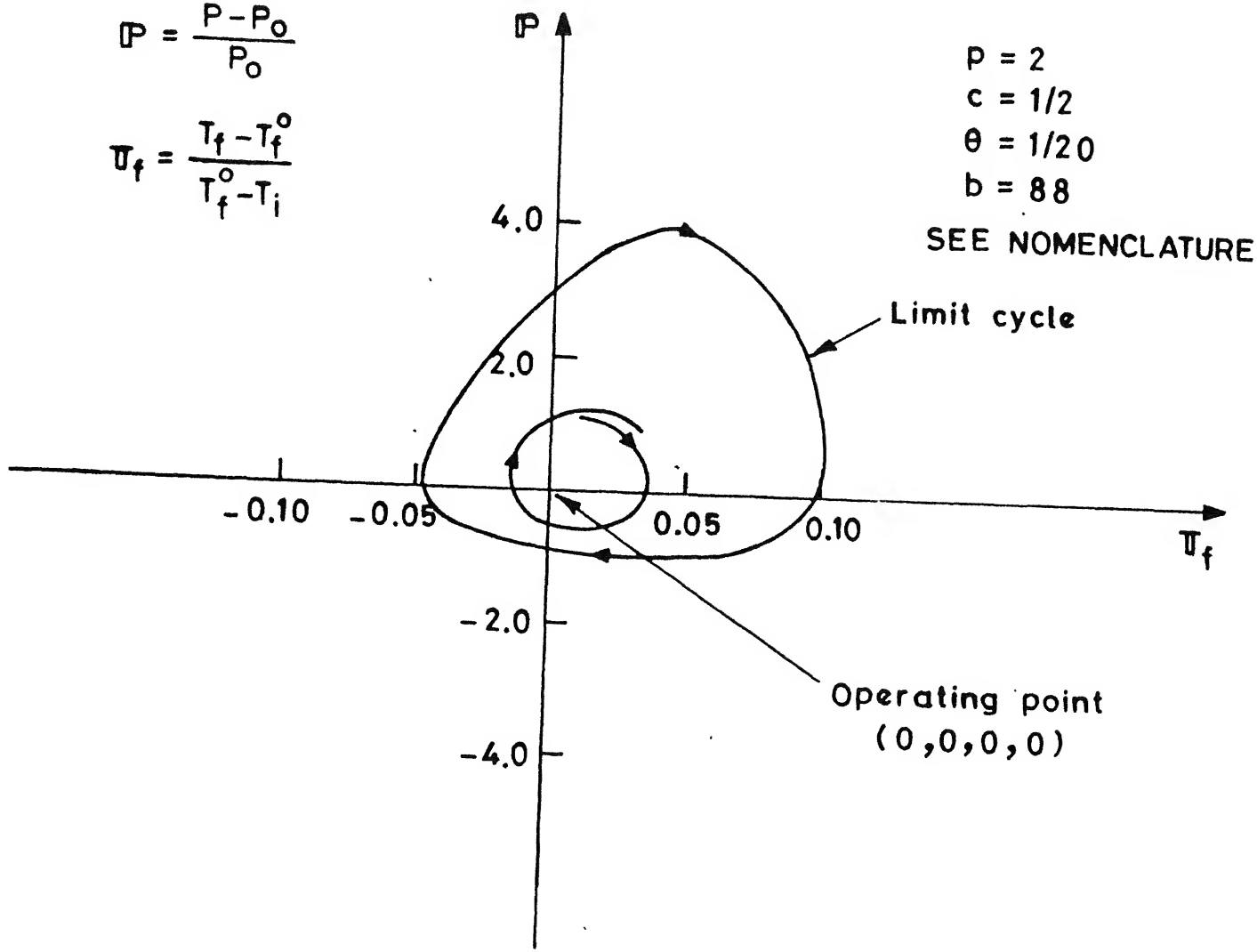


FIG. 9 PROJECTION OF TRANSIENT ORBIT ON $\mathbb{P}$ Vs. $\mathbb{T}_f$ PLANE WITH DELAYED NEUTRONS (Negative fuel and coolant reactivity feedback ; $a_f = -350$, $a_c = -8500$).

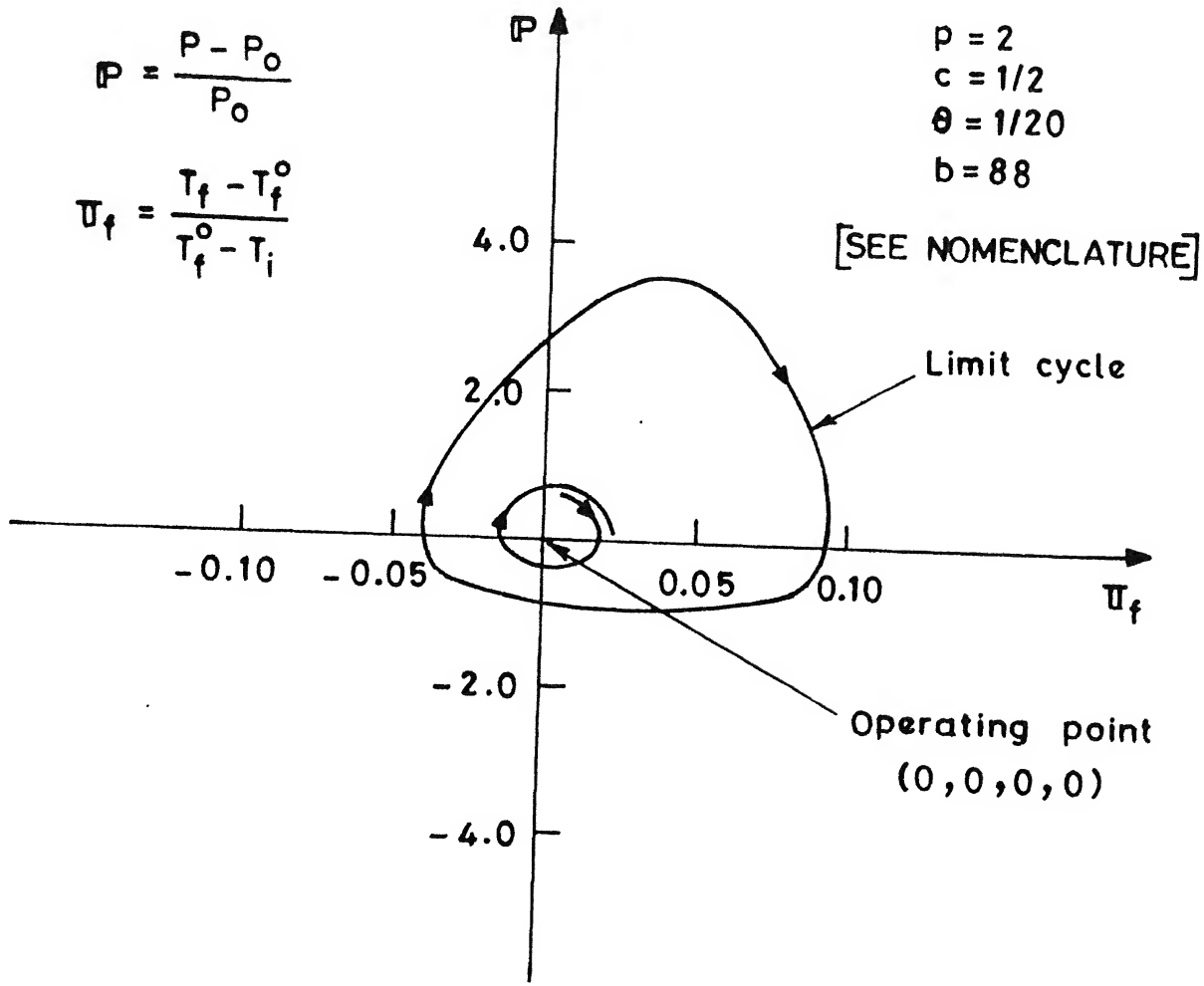


FIG. 10 PROJECTION OF A TRANSIENT ORBIT ON π_f Vs. $\mathbb{P}$ PLANE WITH DELAYED NEUTRONS (Negative fuel and coolant reactivity feedback ; $a_f = -460$, $a_c = -9000$)

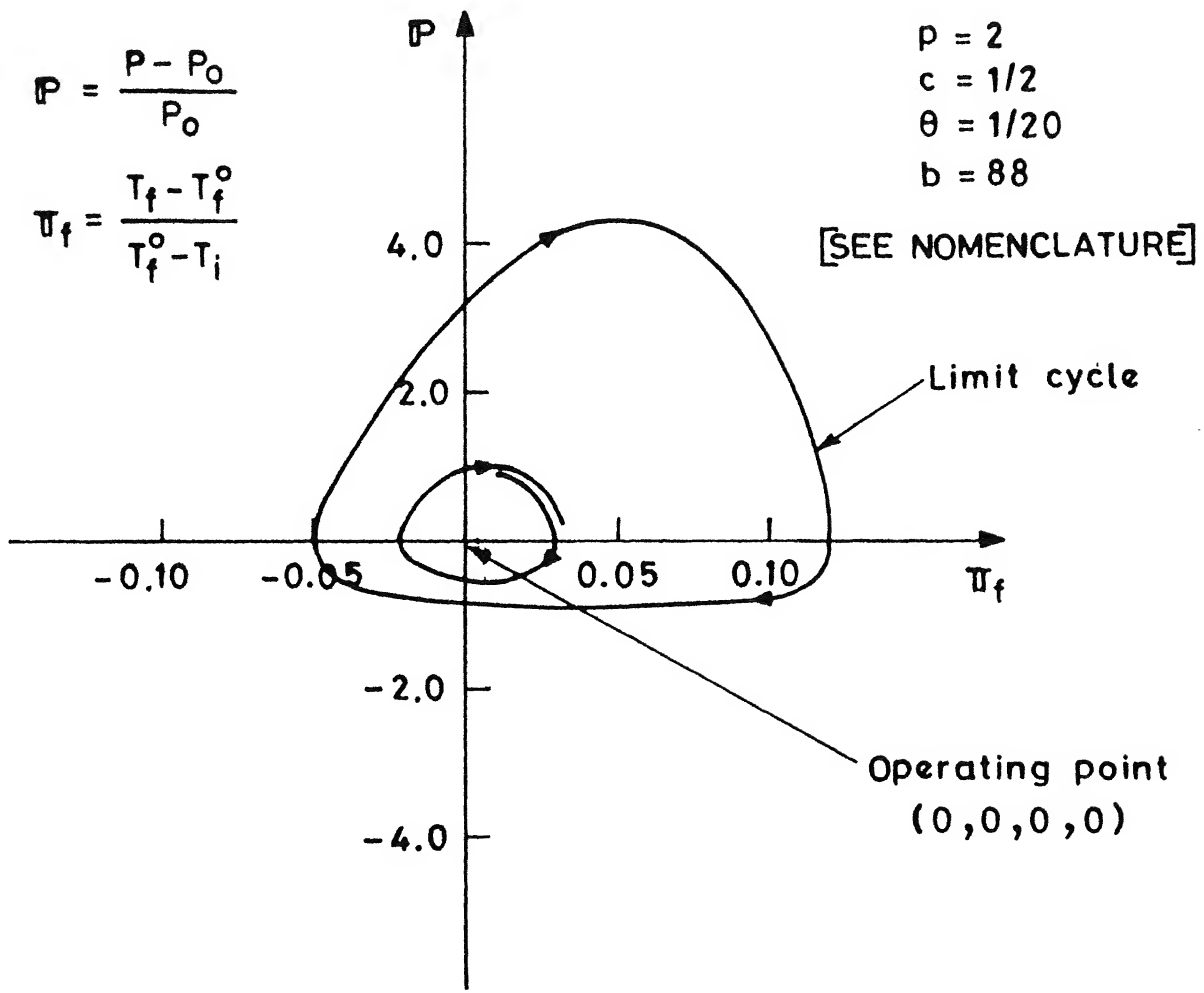


FIG. 11 PROJECTION OF A TRANSIENT ORBIT ON π_f Vs. $\mathbb{P}$ PLANE WITH DELAYED NEUTRONS (Negative fuel and coolant reactivity feedback ; $a_f = -240$, $a_c = -8000$)

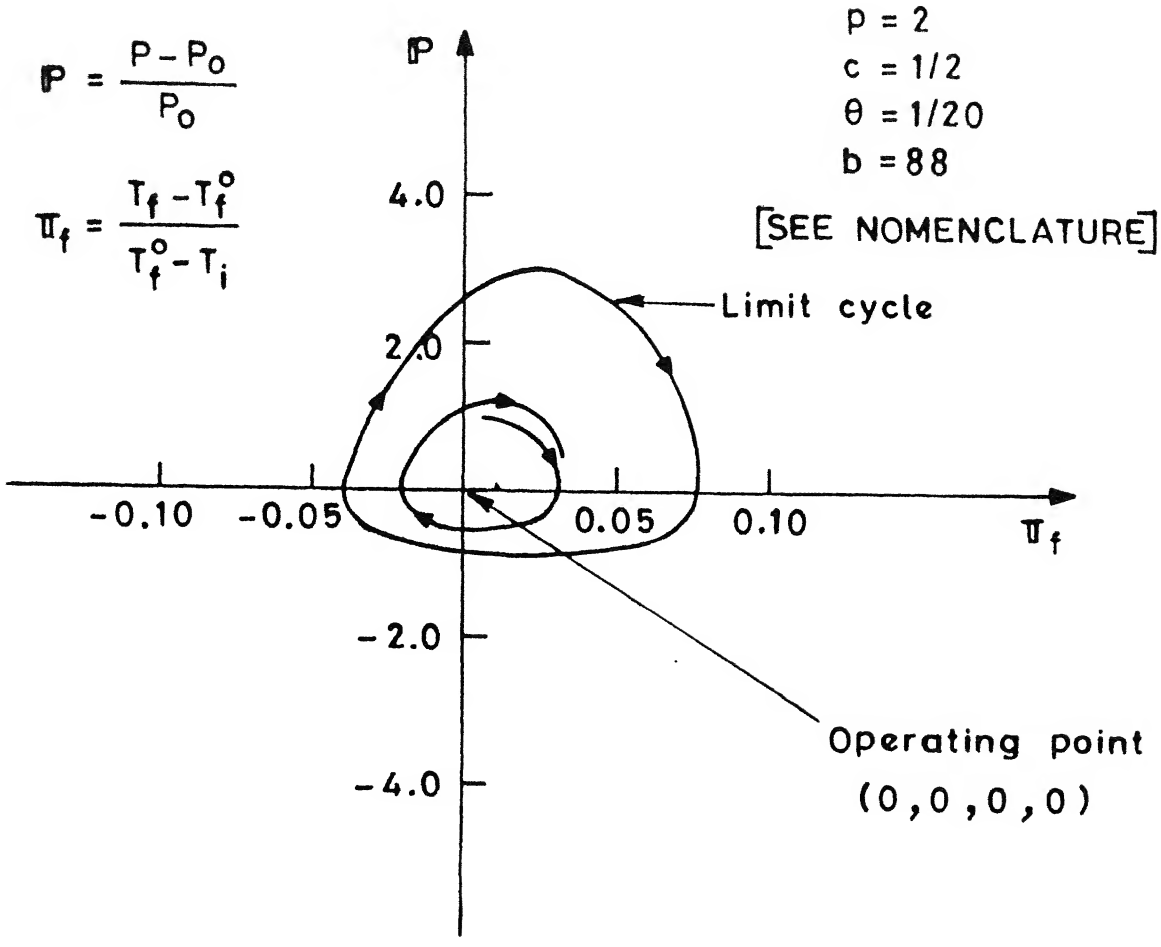


FIG. 12 PROJECTION OF A *TRANSIENT ORBIT*

ON π_f Vs. $\mathbb{P}$ PLANE WITH DELAYED NEUTRONS
(Negative fuel and coolant reactivity feedback ;
$a_f = -680$, $a_c = -10000$)

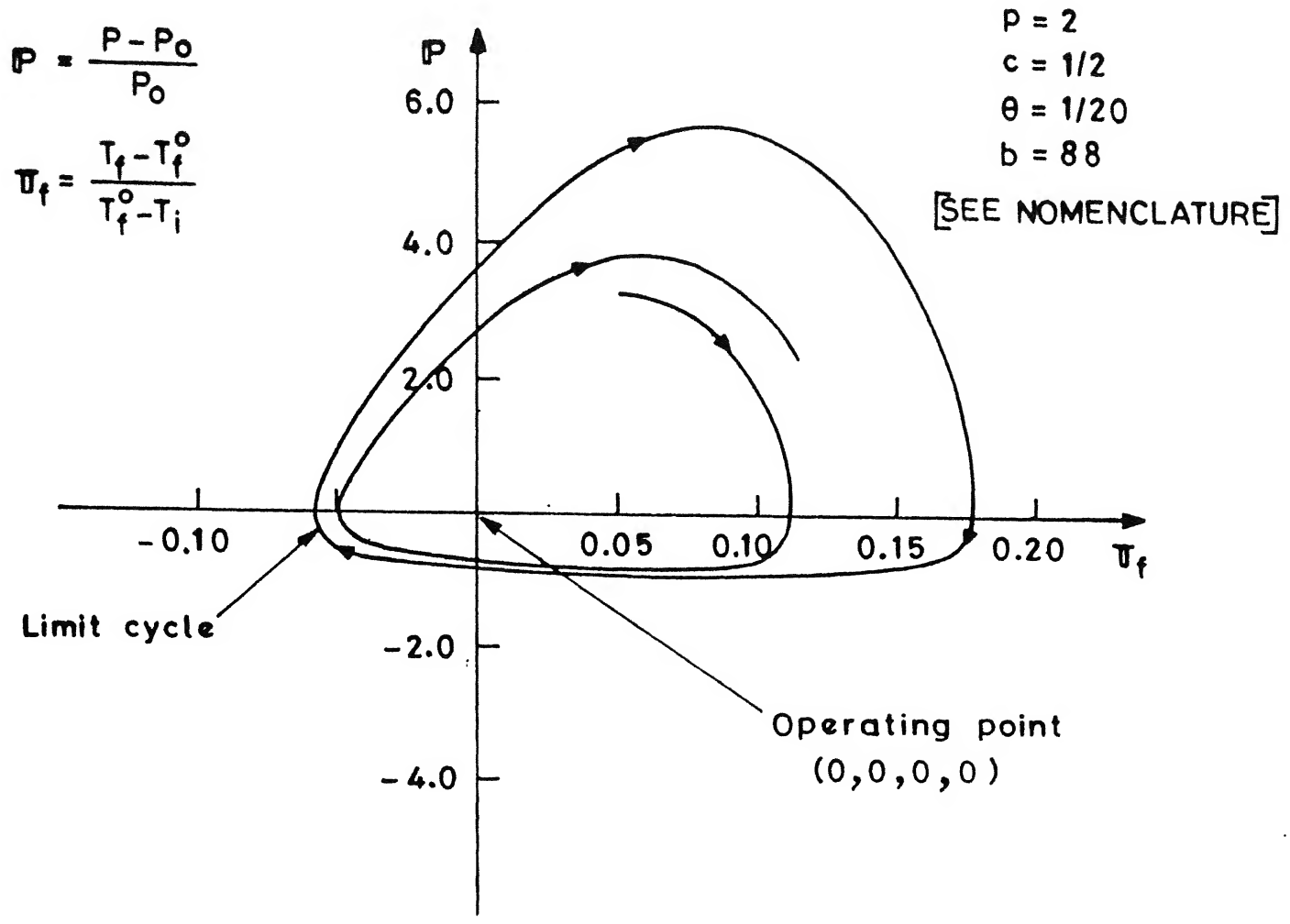


FIG. 13 PROJECTION OF A TRANSIENT ORBIT ON $\mathbb{T}_f$ Vs. $\mathbb{P}$ PLANE WITH DELAYED NEUTRONS

(Fuel reactivity feedback positive and coolant reactivity feedback negative ; $a_f = 89$, $a_c = -6500$)

able 1: Summary of Time Periods and Percent Fluctuations in Power and Fuel Temperature for the Limit Cycles Analyzed.

CASE 1. Without Delayed Neutrons

xp. o.	a_c	a_f	Time Periods (s)	Percent Power Fluctuation	Percent Fuel Temperature Fluctuation	Limit Cycle Plotted in Fig. #
1	-30	-77	6.26	67	8.0	4
2	-27	-47	7.46	64	9.0	5
3	-50	-275	3.60	81	5.4	6

ɔle 2: Summary of Time Periods and Percent Fluctuations in Power and Fuel Temperature for the Limit Cycles Analyzed.

CASE 2. With Delayed Neutrons

p.	a_c	a_f	Time Periods (s)	Percent Power Fluctuation	Percent Fuel Temperature Fluctuation	Limit Cycle Plotted in Fig. #
	-8500	-350	2.09	400	10.0	9
	-9000	-460	1.99	350	9.5	10
	-8000	-240	2.23	430	12.0	11
	-10000	-680	1.80	300	7.5	12
	-6500	89	2.77	570	18.0	13

Table 3: Typical Values for the Reactor Parameters Used in the Present Study (Generally Appropriate for Thermal Reactors).

Parameters	Typical Values
β	0.0066
λ	0.075 s^{-1}
T_f^o	1250 K
T_c^o	560 K
T_i^o	525 K
l	$2x10^{-3}$ - $5x10^{-5}$ s
P_o	1000 MW
$\lvert\alpha_f\rvert$	10^{-4} - 10^{-6}
$\lvert\alpha_c\rvert$	10^{-3} - 10^{-6}
C_f	10^7 J K^{-1}
C_c	2 x 10^7 J K^{-1}

REFERENCES

[1] J.P. Keener and D.S. Cohen, "Nonlinear Oscillation in a Reactor with two Temperature Coefficients", Nucl. Sci. & Engg., 56, 354-359 (1975).

[2] A.Z. Akcasu and L.M. Shotkin, "Power Oscillation and Describing Function in Reactors with Linear Feedback", Nucl. Sci. & Engg., 28, 72-81 (1967).

[3] L.M. Shotkin, D.L. Hetrick and T.R. Schmidt, "Effect of Delayed Neutrons on Autonomous Nonlinear Power Oscillations," Nucl. Sci. & Engg., 42, 10-15 (1970).

[4] T.R. Schmidt and D.L. Hetrick, "Nonlinear Oscillation and Stability of a Nuclear Reactor with Two Reactivity Feedbacks," Nucl. Sci. & Engg., 42, 1-9 (1970).

[5] R.S. Poddar, M.S. Trasi and V. Balaraman, "Power Oscillation in a Reactor with two Temperature Coefficients," Atomkernenergie Kerntechnik, Vol. 42, (1983), No. 2.

[6] M.E. Ward and John, C. Lee, "Singular Perturbation Analysis of Relaxation Oscillation in Reactor System," Nucl. Sci. & Engg., 95, 47-59 (1987).

[7] J. Devooght and H.B. Smets, "Determination of Stability Domains in Point Reactor Dynamics", Nucl. Sci. & Engg., 28, 226-236 (1967).

[8] S.A. Vreeke and G.M. Sandquist, "Phase Space Analysis of Reactor Kinetics", Nucl. Sci. & Engg., 42, 295-305 (1970).

[9] R.S. Poddar and M.S. Trasi, "Temperature Coefficients for Global Stability in a Reactor with Two-Temperature Feedback", J. Appl. Maths. & Phys., Vol. 34, March, 1983.

[10] H.P. Gupta and M.S. Trasi, "Asymptotically Stable Solutions of Point-Reactor Kinetics Equations in the Presence of Newtonian temperature Feedback", Nucl. Energy, Vol. 13, No. 4, pp. 203-207, 1986.

[11] M.E. Ward and John, C. Lee., "Singular Perturbation Analysis of Limit Cycle Behavior in Nuclear - Coupled Density - Wave Oscillations", Nucl. Sci. & Engg., 97, 190-202, (1987).

[12] D.S. Cohen and J.P. Keener, "Oscillatory Processes in the Theory of Particulate Formation in Supersaturated Chemical Solutions", SIAM J. Appl. Math., Vol. 28, No. 2, March, 1975.

[13] Koncay Huseyin, "Multiple Parameter Stability Theory and its Applications", Oxford Engineering Science Series 18, Clarendon Press.

[14] M. Ash, "Nuclear Reactor Kinetics", McGraw-Hill.

[15] J.E. Marsden and M. McCracken, "The Hopf Bifurcation and its Applications", Springer-Verlag (1976).

[16] J.A. Richards, "Analysis of Periodically time-Varying Systems", Springer-Verlag (1983).

[17] Ashok Kumar, "Two Temperature Reactor Model with Delayed Neutrons and Feedback Effects", M.Tech. Thesis, Nuclear Engg. & Technology Programme., IIT Kanpur, India (1984).

[18] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, "Theory and Applications of Hopf Bifurcation", Cambridge Univ. Press (1981).

[19] J. R. Lamarsh, "Introduction to Nuclear Enginreering, Addison-Wesley Publishing Company, Inc., 1983.

[20] K. Ogata, "Modern Control Engineering", Prentice Hall, (1970).

[21] P. Antman and J. Keller, "Bifurcation theory and Nonlinear Eigen value Probleme", Benjamin (1969).

[22] W. Baran and K. Meyer, "Effect of Delayed Neutrons on the Stability of a Nuclear Power Reactor", Nucl. Sci. & Engg., 24, 356-361 (1966).

[23] H.B. Smets, "On the Effect of Delayed Neutrons in Reactor Dynamics", Nucl. Sci. & Engg., 25, 236-241 (1966).

[24] Manmohan Pandey, "Hopf Bifurcation and Limit Cycles in Fission Reactor Dynamics", M.Tech. Thesis, Dept. of Mechanical Engineering, IIT Kanpur, India (1988).

[25] V.M. Popov, "Notes on the Inherent Stability of Nuclear Reactors", Intern. Conf. Peaceful Uses Atomic Energy", Geneva, 2nd Conf. p. 2458 (1958).

[26] "CRC Handbook of Nuclear Reactors Calculations", Edited by Yigal Ronen, CRC Press (1986).

[27] S. Glasstone and M.C. Edlund, "The Elements of Nuclear Reactor Theory", D. Van Nostrand, New York, 1952.

[28] J.R. Lamarsh, "Introduction to Nuclear Reactor Theory", Addison-Wesley, Reading, Mass., (1966).

[29] R. Devaney, "Introduction to Chaotic Dynamical Systems", Benjamin Cummings (1986).

[30] D.L. Hetrick, "Dynamics of Nuclear Reactors", Univ. of Chicago Press (1971).

[31] B.C. Kuo, "Automatic Control Systems", Prentice Hall of India Pvt. Ltd., (1982).

[32] E.P. Gyftopoulos and J. Devooght, "Effect of Delayed Neutrons on Nonlinear Reactor Stability", Nucl. Sci. & Engg., 8, 244-250 (1960).

[33] L.M. Shotkin, "A Nonlinear Analysis of a Reactor with Two-Temperature Coefficients", Nucl. Sci. & Engg., 18, 271-279 (1964).

[34] G.S. Lellouche, "Reactor-Kinetics Stability Criteria", Nucl. Sci. & Engg., 24, p. 72 (1966).

[35] P. Henrici, "Discrete Variable Methods in Ordinary Differential Equations, Wiley, New York, 1962, Ch. 5.

[36] C.W. Gear, "The Automatic Integration of Ordinary Differential Equations", Comm. of the ACM, 14, p. 176 (1971).

[37] G. Hall and J.M. Watt, "Modern Numerical Methods for Ordinary Differential Equations", Clarendon Press, Oxford (1976).

[38] C.W. Gear, "The Automatic Integration of Stiff Ordinary Differential Equations", Information Processing, 68, A.J.H.

[39] M. Kubicek and M. Marek, "Computational Methods in Bifurcation Theory and Dissipative Structures", Springer-Verlag, New York Inc., (1983).

A107894

Th
621.4343
Si64b

Date Slip

This book is to be returned on the date last stamped.

NETP-1989-M-SIN-POW